The Fifth Clemson mini-Conference

ON[R]

Discrete Mathematics

Clemson, South Carolina
October 11-12, 1990

DEPARTMENT
OF
MATHEMATICAL
SCIENCES

CLEMSON UNIVERSITY
Clemson, South Carolina

CLEMSON
UNIVERSITY

Best
Available
Copy

Unclassified

CURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION Unclassified	1b. RESTRICTIVE MARKINGS None
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION / AVAILABILITY OF REPORT Unclassified/unrestricted
2b. DECLASSIFICATION / DOWNGRADING SCHEDULE NA	
4. PERFORMING ORGANIZATION REPORT NUMBER(S) None	5. MONITORING ORGANIZATION REPORT NUMBER(S) None
6a. NAME OF PERFORMING ORGANIZATION Clemson University — 6b OFFICE SYMBOL (If applicable) NA	7a. NAME OF MONITORING ORGANIZATION Office of Naval Research Resident Representative N66020
6c. ADDRESS (City, State, and ZIP Code) Clemson, SC 29634-5355	7b. ADDRESS (City, State, and ZIP Code) Administrative Contracting Officer Georgia Inst. of Technology 206 O'Keefe Building Atlanta, GA 30332-0490
8a. NAME OF FUNDING / SPONSORING ORGANIZATION Dept. of the Navy — 8b. OFFICE SYMBOL (If applicable) 1511:MAW	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
8c. ADDRESS (City, State, and ZIP Code) 800 N. Quincy Street Arlington, VA 22217-5000	10. SOURCE OF FUNDING NUMBERS PROGRAM ELEMENT NO. / PROJECT NO. / TASK NO. / WORK UNIT ACCESSION NO.

11. TITLE (Include Security Classification)
The Fifth Clemson mini-Conference ON[R] Discrete Mathematics

12. PERSONAL AUTHOR(S)
S. T. Hedetniemi and R. Laskar

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Year, Month, Day)	15. PAGE COUNT
Proceedings	FROM 10-11-90 TO 9-30-91	1991 July 10	262

16. SUPPLEMENTARY NOTATION

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	Discrete Mathematics

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

This contract provided funds to partially support the Clemson Mini-Conference ON[R] Discrete Mathematics (5th annual). This two day conference featured twelve speakers from the following colleges and universities: the Georgia Institute of Technology, Northeastern University, the College of William and Mary, Memphis State University, the University of Illinois (2), Ohio State University, the University of Tennessee, Wright State University, Vanderbilt University(2) and Old Dominion University. There were approximately 80 attendees. The conference has been sponsored by the Office of Naval Research for five years and in that time the conference has attracted most of the leading researchers in graph theory and discrete mathematics in the United States and some international visitors. The funds were used to pay part of the expenses of of the speakers, for publication of the proceedings of the conference and for a small reception given during the conference.

20. DISTRIBUTION / AVAILABILITY OF ABSTRACT ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT. ☐ DTIC USERS	21. ABSTRACT SECURITY CLASSIFICATION Unclassfied
22a NAME OF RESPONSIBLE INDIVIDUAL E. Chris Thurston	22b. TELEPHONE (Include Area Code) (803) 656-0636 — 22c. OFFICE SYMBOL

DD FORM 1473, 84 MAR — 83 APR edition may be used until exhausted. All other editions are obsolete.

SECURITY CLASSIFICATION OF THIS PAGE
Unclassified

The Fifth Clemson mini-Conference

ON[R]

Discrete Mathematics

Clemson, South Carolina
October 11-12, 1990

Organizers: S. T. Hedetniemi
R. Laskar

A-1

This mini-Conference was supported in part by the Office of Naval Research for the University Research Initiative Program, Contract NO. N00014-91-K-1228.

91

The Fifth Clemson mini-Conference

ON[R]

Discrete Mathematics

Clemson, South Carolina
October 11-12, 1990

Schedule of talks
(All talks given in Student Senate Chambers)

Thursday, October 11

11:00 - 12 noon **_Registration_**

1:00 - 1:10 Welcoming Remarks by Dr. Bobby Wixson, Dean of College of Sciences

1:10 - 1:50 *Prof. Richard A. Duke, Department of Mathematics Georgia Tech.*

"The Erdős-Ko-Rado Theorem for Small Families"

Let X be a set of size n, $\mathcal{F}$ a family of m k-element subsets of X, $k < n/2$, and $\mathcal{F}'$ a subfamily of $\mathcal{F}$ with the property that $|F_1 \cap F_2| \geq t$ for each choice of F_1 and F_2 in $\mathcal{F}'$. It follows immediately from the well-known Erdös-Ko-Rado Theorem that for m near $\binom{n}{k}$ and n sufficiently large the maximum size of $\mathcal{F}'$ in this case is of order $(k/n)^t m$.

In general let $f_t(n,k,m)$ be the minimum of $\max\{|\mathcal{F}'|: \mathcal{F}' \subseteq \mathcal{F}, |F_1 \cap F_2| \geq t$ for each F_1 and F_2 in $\mathcal{F}'\}$, over all families $\mathcal{F}$ of k-element subsets of X, $|\mathcal{F}| = m$, $|X| = n$. Then for n large and m near $\binom{n}{k}$ we have $f_t(n,k,m) \sim (k/n)^t m$. In joint work with V. Rödl we investigate this function for small m. We show, for example, that if $k = cn$, $0 < c < \frac{1}{2}$, and $m = n$, then $f_2(n,k,m) \sim cn = (k/n)m$ (instead of $(k/n)^2 m$ as might be expected). Our proof makes use of the Regularity Lemma of Szemerédi. Taking $\mathcal{F}$ to be the collection of lines of a finite protective plane shows that $k = cn$ cannot be replaced by $k = \sqrt{n}$ in this result. In fact, we show that cn cannot be replaced by $\sqrt{n}\ \mathrm{1n}\ (n)$. The case of $m = n$ and $k = n^{1-\epsilon}$, $0 < \epsilon < \frac{1}{2}$, remains open.

2:10 - 2:50

Prof. Margaret B. Cozzens, Department of Mathematics
Northeastern University

"Critical m-neighbor-connected Graphs"

Let G be a graph and u be a vertex in G. The closed neighborhood of u is N[u] = {u} ∪ N(u). A vertex u is *subverted* when N[u] is deleted from G. If S is a subset of vertices of G, then G/S denotes G - {N[u]: u ε S}. S is called a *cut-strategy* of G if G/S is disconnected, or a clique, or the empty set. We define the neighborhood-connectivity, K(G), to be the minimum size of all cut-strategies S of G. A graph G is said to be *m-neighbor-connected* if K(G) = m, and *critically m-neighbor connected* if K(G) = m and for any vertex v, K(G/{v}) = m - 1. Gunther in 1978 and 1985 modeled the reliability of a spy network using the neighbor-connectivity of a graph.

A graph G is a *minimum critically m-neighbor-connected* graph if no critically m-neighbor-connected graph with the same number of vertices has fewer edges than G. Cozzens and Wu give upper bounds on the minimum size of the critically m-nieghbor-connected graphs of fixed order v and show that the number of edges in a minimum critically m-neighbor-connected graph with order v , where v is a multiple of m, is $\left\lceil \frac{mv}{2} \right\rceil$, hence such a graph is always m-regular.

Examples of m-neighbor connected graphs and methods of constructing m-neighbor-connected graphs will be given in this talk. Insight into the structure of this class of graphs will be provided. There are many open problems relating the parameter K to other parameters of connectedness, and domination. These will be discussed.

3:10 - 3:50

Prof. Douglas R. Shier, Department of Mathematics
College of William and Mary

"Cancellation and Consecutive Sets"

The principle of inclusion and exclusion has been applied to numerous areas of discrete mathematics. One manifestation of this principle occurs in expressing the probability of the union of events in terms of the alternating sum of probabilities of intersections of events. If the events themselves are sufficiently well structured, then predictable cancellation occurs in this expansion. This talk discusses the special case of "consecutive sets," in which elements occur consecutively in every set. For such sets the inclusion-exclusion expansion assumes a particularly nice form, with all reduced coefficients being ±1. In fact the appropriate sign is determined by the length of a certain path in a graph derived from the incidence structure of the given sets.

4:10 - 4:50

Prof. Richard H. Schelp, Department of Mathematical Sciences
Memphis State University

Andrew Sobczyk Memorial Lecture

"The Local Ramsey Number and Local Colorings"

A local k-coloring of a graph H is a coloring of the edges of H (by any number of colors) in such a way that the edges incident to each vertex of H are colored with at most k different colors. The local Ramsey number $r^k_{loc}(G)$ is defined as the smallest positive integer m such that K_m contains a monochromatic copy of G for every local k-coloring of K_m. This Ramsey number exists and is at least as large as the usual Ramsey number $r^k(G)$ of G for k colors Results and open questions will be presented for the local Ramsey number as well as for a generalization of local k-colorings.

7:30

Social, Jordan Room

Friday, October 12

8:00

Coffee and Doughnuts, Student Senate Chambers

8:10 - 8:50

Prof. Pravin Vaidya, Department of Computer Science
University of Illinois

"New algorithms for minimizing convex functions over convex sets"

Let $S \subseteq R^n$ be a convex set for which there is an oracle with the following property. Given any point $z \in R^n$ the oracle returns a "Yes" if $z \in S$; whereas if $z \notin S$ then the oracle returns a "No" together with a hyperplane that separates z from S. The *feasibility problem* is the problem of finding a point in S; the *convex optimization problem* is the problem of minimizing a convex function over S. We present a new class of algorithms for the feasibility problem based on enclosing the target set S in a polytope whose volume shrinks geometrically at each step. A suitable center of the current polytope is used as a test point at each step. The new algorithms are faster than the previously best known algorithms by a factor proportional to n. The algorithms for the feasibility problem easily adapt to the convex optimization problem.

9:10 - 9:50 *Prof. Dijen K. Ray-Chaudhuri, Department of Mathematics*
Ohio State University

"Size of an s-intersection family in a semilattice and construction of vector space designs by quadratic forms"

Let v, k and s be positive integers, $v \geq k + s$. Let X be a v-set and $\mathcal{Q}$ be a set of subsets of X, each subset containing k elements. $\mathcal{Q}$ is called a k-uniform s-intersection family if and only if $|\{|A \cup B|; A, B \in \mathcal{Q}, A \neq B\}| = s$. Ray-Chaudhuri and Wilson in their 1975 paper proved that if $\mathcal{Q}$ is a k-uniform s-intersection family of v-set X, then $|\mathcal{Q}| \leq \binom{v}{s}$. This theorem is generalized to a class of semilattices called polynomial semilattices which include many important combinatorial structures. Let V be a v-dimensional vector space over a finite field of order q and $\mathcal{B}$ be a family of k-dimensional subspaces of $\mathcal{B}$. The pair $(V, \mathcal{B})$ is called a t - $[v, k, \lambda, q]$ design iff every t-dimensional subspace T of V is contained in exactly λ elements B of $\mathcal{B}$. We construct several families of vector space designs for $t = 2$ and 3 by using quadratic forms.

10:10 - 10:50 *Prof. Douglas B. West, Department of Mathematics*
University of Illinois-Urbana

"A Graph-theoretic Game and Its Application to the k-Server Problem"

We consider a zero-sum game played on the graph between a tree player and an edge player. The tree player chooses a spanning tree T and the edge player chooses an edge e. If e lies in T then the payoff to the edge player is zero; otherwise, the payoff is the length of the unique cycle created when e is added to T. We determine the value of the game for specific classes of graphs and derive an upper bound on the value for any n-vertex graph. These results yield new competitive randomized algorithms for the k-server problem on a wide class of metric spaces. For example, we obtain a 2k-competitive algorithm (against oblivious adversaries) for the k-server problem on a circle. This is joint work with Noga Alon and Richard Karp.

11:10 - 11:50 *Prof. Michael Langston, Department of Computer Science*
University of Tennessee

"Polynomial-Time Algorithms from Finite Basis Theorems - A Survey"

Traditionally, problems have been roughly classified as either "easy" or "hard", dependent on whether low-degree, polynomial-time, decision algorithms exist to solve them. Until recently, one could expect any proof of easiness to be constructive. That is, the proof itself should provide positive evidence in the form of the promised polynomial-time algorithm.

This appealing picture is dramatically altered, however, by recent "nonconstructive" developments in the theory of well-partially-ordered sets. New algorithmic characterizations are now possible that rely on finite but unknown bases of forbidden subgraphs.

In this talk we will survey some of the main results and open questions related to this general topic.

LUNCH

1:30 - 2:10

Prof. Gerd H. Fricke, Department of Mathematics and Statistics
Wright State University

"On the Product of the Independent Domination Numbers of a Graph and Its Complement"

Let i(G) denote the smallest cardinality of an independent dominating set (equivalently maximal independent set) of vertices of a graph G. We will study $m\bar{i}\bar{i}(p) = \max_{|G|=p} i(G)i(\bar{G})$, the maximum value over p vertex graphs of the product of the independent domination numbers of a graph and its complement. Recently Cockayne, Favoron, Li, and MacGillivray have shown that $i(G)i(\bar{G}) \leq \min\left\{\frac{(p+3)^2}{8}, \frac{(p+8)^2}{10.8}\right\}$. We will show that $m\bar{i}\bar{i}(p)$ behaves like $\frac{p^2}{16}$ asymptotically by proving the following:

Theorem: Let $0 < k < 16$. Then there exists an integer p_0 such that
$$i(G)i(\bar{G}) \leq \frac{p^2}{k} \text{ for any graph } G \text{ with } |G| = p \geq p_0 .$$

2:20 - 3:00

Prof. Jeremy Spinrad, Department of Computer Science
Vanderbilt University

"Containment of Circular-Arcs"

The neighborhood containment matrix of an n vertex graph is an n by n matrix M such that M[x,y] = 1 exactly when N(x) - {y} contains N(y) - {x}. This talk presents a method for determining the neighborhood containment matrix of a circular-arc graph in $O(n^2)$ time. Computing the neighborhood containment matrix was a bottleneck step, and possibly the only bottleneck step, of Tucker's recognition algorithm for circular-arc graphs. The techniques for computing this matrix involve reduction of the problem to containment problems on chordal bipartite graphs, and using special properties of chordal bipartite graphs. We also pose several open problems on chordal bipartite graphs.

3:10 - 3:50 *Prof. Stephan Olariu, Department of Computer Science*
Old Dominion University

"A Fast Parallel Recognition Algorithm for a Class of Tree-representable Graphs"

A number of problems in computational semantics, group-based cooperation, networking, examination scheduling, to name just a few, suggested the study of graphs featuring certain "local density" characteristics. Typically, the notion of local density is equated with the absence of chordless paths of length three or more. Recently, a new metric for local density has been proposed, allowing a number of such induced paths to occur. More precisely, a graphs G is *P_4-sparse* if no set of five vertices in G induces more than one chordless path of length three. *P_4-sparse* graphs generalize the well-known class of cographs corresponding to a more stringent local density metric. One remarkable feature of *P_4-sparse* graphs is that they admit a tree representation unique up to isomorphism. In this work we present a parallel algorithm to recognize *P_4-sparse* graphs and show how the data structures returned by the recognition algorithm can be used to construct the corresponding tree representation. With a graph $G = (V,E)$ with $|V| = n$ and $|E| = m$ as input, our algorithms run in $O(\log n)$ time using $O\left(\frac{n^2 + mn}{\log n}\right)$ processors in the EREW-PRAM model.

4:00 - 4:40 *Prof. Mark Ellingham, Department of Mathematics*
Vanderbilt University

"Vertex-switching reconstruction and pseudosimilarity"

A *vertex-switching* G_v of a graph G at a vertex v is obtained by deleting all edges incident with v, and then adding all possible edges incident with v which were not in G. A graph is *vertex-switching reconstructible* if it is determined by its collection of vertex-switchings. Two vertices u and v of G are *vertex-switching pseudosimilar* if they are not similar but G_u and G_v are isomorphic. We talk about some recent advances in the theory of vertex-switching reconstruction, including results on vertex-switching reconstruction of classes of graphs and a characterization of vertex-switching pseudosimilar vertices.

THE ERDÖS-KO-RADO THEOREM FOR SMALL FAMILIES

Prof. Richard A. Duke
Department of Mathematics
Georgia Tech. University

THE ERDÖS-KO-RADO THEOREM FOR SMALL FAMILIES

R. DUKE , V. RÖDL

LET $[n] = \{1, 2, \ldots, n\}$ AND

$$[n]^k = \{A : A \subseteq [n],\ |A| = k\} \qquad (k < n/2)$$

CALL $\mathcal{a}^t \subseteq [n]^k$ <u>t-INTERSECTING</u> IF

$$|A_i \cap A_j| \geq t \quad \text{FOR EACH } A_i, A_j \in \mathcal{a}^t.$$

HOW LARGE CAN SUCH A t-INTERSECTING FAMILY BE?

CHOOSING $\mathcal{a}^t = \{A : A \in [n]^k, \{1, 2, \ldots, t\} \subseteq A\}$ SHOWS THAT

$$\max |\mathcal{a}^t| \geq \binom{n-t}{k-t}.$$

<u>THEOREM</u> (ERDÖS-KO-RADO) GIVEN $k \geq t > 0$ THERE EXISTS $n_0 = n_0(k,t)$ SUCH THAT FOR $n \geq n_0$ IF $\mathcal{a}^t$ IS A t-INTERSECTING FAMILY, $\mathcal{a}^t \subseteq [n]^k$, THEN

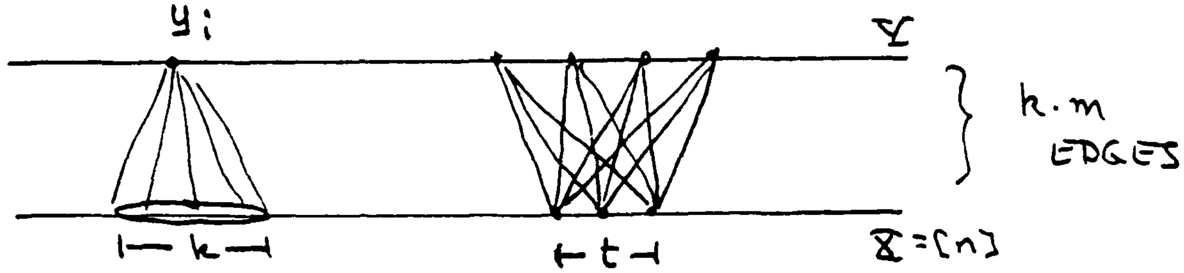
$$|\mathcal{a}^t| \leq \binom{n-t}{k-t}.$$

($n_0(k,t) = (k-t+1)(t+1)$, FRANKL, 1978; WILSON, 1984)

SINCE $\binom{n-t}{k-t} \sim \left(\frac{k}{n}\right)^t \binom{n}{k}$, IF $\mathcal{A} \subseteq [n]^k$, $|\mathcal{A}| = m \sim \binom{n}{k}$ AND $\mathcal{A}^t \subseteq \mathcal{A}$ IS t-INTERSECTING,

$$\max \frac{|\mathcal{A}^t|}{|\mathcal{A}|} \sim \left(k/n\right)^t .$$

LET $\mathcal{A} = \{A_i\}_{i=1}^m$ BE A SUBFAMILY OF $[n]^k$. CONSIDER THE BIPARTITE GRAPH G WITH VERTEX CLASSES $X = [n]$, $Y = \{y_1, y_2, \dots, y_n\}$ IN WHICH $\{l, y_i\}$ IS AN EDGE IFF $l \in A_i$.



SIMPLE AVERAGING SHOWS THAT THERE ARE t VERTICES IN X ALL JOINED TO THE SAME q VERTICES IN Y, WHERE

$$q \geq \binom{k}{t} \Big/ \binom{n}{t} \cdot m \sim \left(k/n\right)^t m .$$

SINCE THESE VERTICES OF Y CORRESPOND TO A t-INTERSECTING FAMILY

$$\max \frac{|\mathcal{A}^t|}{|\mathcal{A}|} \geq \frac{q}{m} \sim \left(k/n\right)^t .$$

LET $f_t(n,k,m)$ BE THE MINIMUM OF $\max \frac{|\mathcal{a}^t|}{|\mathcal{a}|}$ FOR $\mathcal{a}^t \subseteq \mathcal{a} \subseteq [n]^k$, $|\mathcal{a}| = m$, $\mathcal{a}^t$ t-INTERSECTING.

BY THE E-K-R THEOREM, FOR $m \sim \binom{n}{k}$ AND n LARGE, WE HAVE

$$f_t(n,k,m) \sim (k/n)^t.$$

FOR ALL m WE HAVE

$$f_t(n,k,m) \geq \frac{8}{m} \sim (k/n)^t.$$

HERE WE CONSIDER THIS FUNCTION FOR SMALL m.

IN PARTICULAR WE HAVE

<u>THEOREM</u> FOR <u>EACH</u> <u>t</u>, IF $m = n$ AND $k = cn$, $0 < c < 1/2$, THEN FOR n SUFFICIENTLY LARGE

$$f_t(n,k,m) \sim c = (k/n).$$

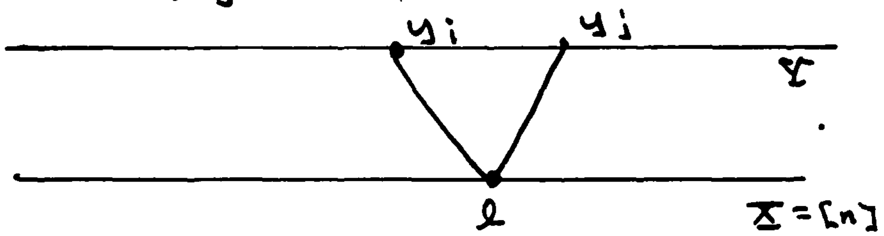
WE SHOW THAT FOR $\mathcal{a} \subseteq [n]^k$, $|\mathcal{a}| = n$, $k = cn$, THERE EXISTS A t-INTERSECTING FAMILY $\mathcal{a}^t \subseteq \mathcal{a}$ WITH $\underline{|\mathcal{a}^t| = cn\,(1 - o(1))}$.

SKETCH OF THE PROOF FOR $\underline{t=2}$.

SUPPOSE $\mathcal{A} = \{A_i\}_{i=1}^n \subseteq [n]^k$, $k = cn$.

CONSIDER THE BIPARTITE GRAPH G AGAIN.
G HAS cn^2 EDGES.

<u>CLAIM</u>. FOR n SUFFICIENTLY LARGE WE CAN DELETE $g(n)$ EDGES FROM G, $g(n) = o(n^2)$, SO THAT IF EDGES $\{\ell, y_i\}$ AND $\{\ell, y_j\}$ REMAIN, THEN $|A_i \cap A_j| \geq 2$.



IF SO, WE ARE DONE!

SINCE THEN SOME VERTEX $\ell \in \mathcal{X}$ STILL HAS DEGREE $\geq \frac{1}{n} \cdot (cn^2 - g(n)) = cn(1 - o(1))$.

ITS NEIGHBORS CORRESPOND TO A 2-INTERSECTING FAMILY.

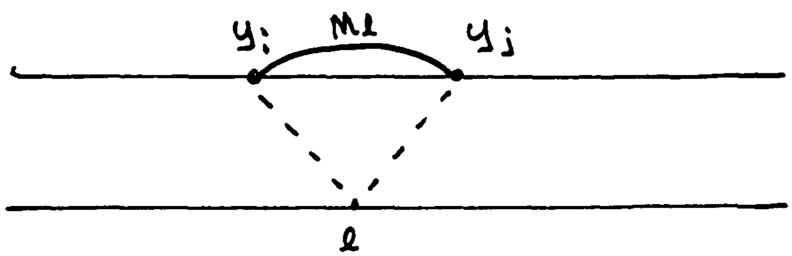
PROOF OF THE CLAIM:

FOR EACH $\ell \in [n]$ FORM A GRAPH G_ℓ WITH VERTEX SET $Y = \{y_1, y_2, \ldots, y_n\}$ IN WHICH $\{y_i, y_j\}$ IS AN EDGE IFF $A_i \cap A_j = \{\ell\}$.

IN EACH G_ℓ CHOOSE A <u>MAXIMAL MATCHING</u>, M_ℓ.

(EACH EDGE OF G_ℓ HAS AN ENDPOINT INCIDENT WITH AN EDGE OF M_ℓ.)

FOR EACH ℓ IF $\{y_i, y_j\}$ IS IN M_ℓ DELETE THE EDGES $\{\ell, y_i\}$ AND $\{\ell, y_j\}$ FROM G.

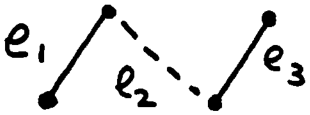


IF $\{\ell, y_i\}$ AND $\{\ell, y_j\}$ REMAIN IN G, THEN $|A_i \cap A_j| \geq 2$.

FOR OTHERWISE, $\{y_i, y_j\}$ IS IN G_ℓ AND AT LEAST ONE OF $\{\ell, y_i\}, \{\ell, y_j\}$ WOULD BE MISSING.

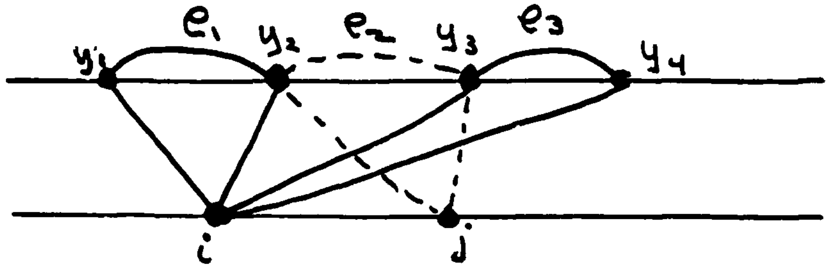
IT REMAINS TO SHOW THAT $\left| \bigcup_{\ell=1}^{n} M_\ell \right| = o(n^2)$.

NOTE THAT IN $\bigcup_{\ell=1}^{n} M_\ell$ WE CAN NOT HAVE



WITH $e_1, e_3 \in M_i$, $e_2 \in M_j$.

THIS WOULD REQUIRE



BUT THEN $i \in A_2 \cap A_3$, SO $A_2 \cap A_3 \neq \{j\}$.

<u>THEOREM</u> (RUZSA, SZEMERÉDI, 1978) FOR m SUFFICIENTLY LARGE IF G IS A BIPARTITE GRAPH WITH $2m$ VERTICES AND cm^2 EDGES WHICH ARE THE UNION OF $\leq m$ MATCHINGS, THEN THERE EXIST MATCHINGS M_i AND M_j AND EDGES $e_1, e_3 \in M_i$, $e_2 \in M_j$, WITH e_2 INCIDENT WITH BOTH e_1 AND e_3.

SINCE $\bigcup_{\ell=1}^{n} M_\ell$ DOES NOT HAVE SUCH EDGES,

$$\left| \bigcup_{\ell=1}^{n} M_\ell \right| = o(n^2).$$

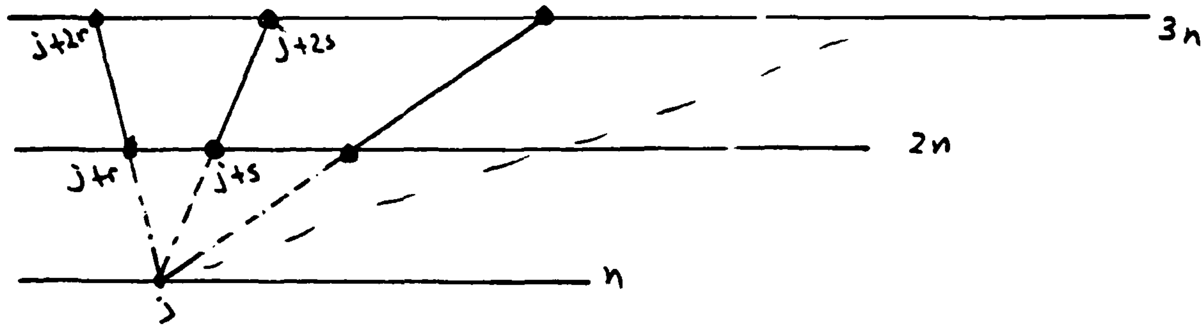
THIS RESULT FOLLOWS FROM SZEMERÉDI'S REGULARITY LEMMA AND WAS USED TO SHOW THE FOLLOWING:

LET $v_3(n) = \max\{|S| : S \subseteq [n],\ S$ DOES <u>NOT</u> CONTAIN A 3-TERM ARITHMETIC PROGRESSION$\}$. THEN

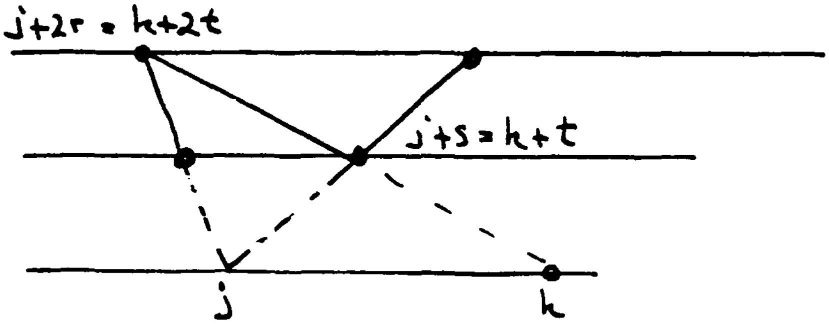
$$v_3(n) = o(n).$$

SUPPOSE $S \subseteq [n]$, WITH $|S| = cn$, $0 < c < 1$.

FOR EACH $j \in [n]$ AND EACH $r \in S$ JOIN $j+r$ TO $j+2r$. THIS YIELDS A MATCHING FOR EACH j ($1 \le j \le n$) IN A BIPARTITE GRAPH WITH $n|S| = cn^2$ EDGES.



THE RUZSA, SZEMERÉDI RESULT INSURES THAT



$$\begin{array}{r} j+2r = k+2t \\ j+s = k+t \\ \hline 2r-s = t \end{array}$$

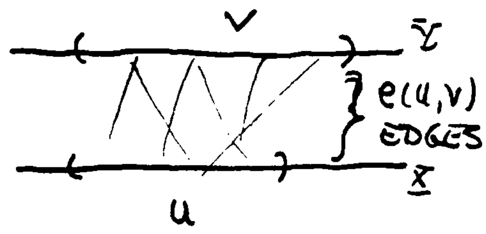
$$r = \frac{s+t}{2}$$

$s,\ r = \frac{s+t}{2},\ t$ FORM A 3-TERM ARITHMETIC PROGRESSION IN S.

SZEMERÉDI'S REGULARITY LEMMA (BIPARTITE VERSION)

FOR A BIPARTITE GRAPH H WITH VERTEX CLASSES X AND Y AND $U \subseteq X$, $V \subseteq Y$:

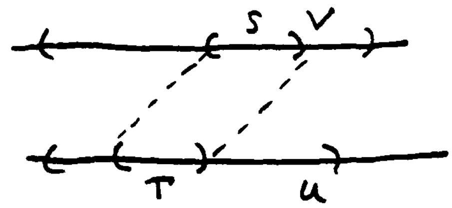
<u>DENSITY</u> $d(U,V) = \dfrac{e(U,V)}{|U|\,|V|}$



<u>ε-REGULAR PAIR</u> (U,V)

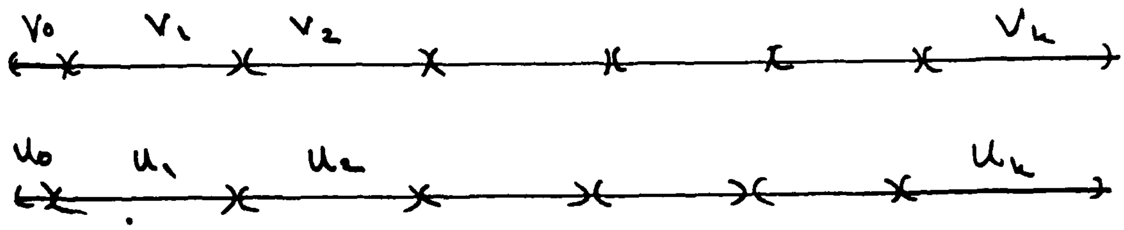
FOR EACH $T \subseteq U$, $S \subseteq V$ WITH $|T| \geq \varepsilon |U|$, $|S| \geq \varepsilon |V|$

$$|d(T,S) - d(U,V)| < \varepsilon$$



SUPPOSE $|X| = |Y| = n$.

<u>THEOREM</u> (SZEMERÉDI) FOR EACH $\varepsilon > 0$, THERE EXIST INTEGERS $N(\varepsilon)$, $K(\varepsilon)$ SUCH THAT FOR $n \geq N(\varepsilon)$ THERE ARE PARTITIONS $X = U_0 \cup U_1 \cup \cdots \cup U_k$ AND $Y = V_0 \cup V_1 \cup \cdots \cup V_k$, $k \leq K(\varepsilon)$, WHERE $|U_0|, |V_0| < \varepsilon n$, $|U_1| = \cdots = |U_k|$, $|V_1| = \cdots = |V_k|$, AND ALL BUT εk^2 OF THE PAIRS (U_i, V_j), $1 \leq i,j \leq k$, ARE ε-REGULAR.



SZEMERÉDI'S REGULARITY LEMMA IMPLIES THE RESULT ON MATCHINGS (IN A BIPARTITE GRAPH).

SUPPOSE G IS A BIPARTITE GRAPH WITH n VERTICES IN EACH CLASS, cn^2 EDGES IN AT MOST n MATCHINGS.

APPLY THE REGULARITY LEMMA (WITH SUITABLE ε).

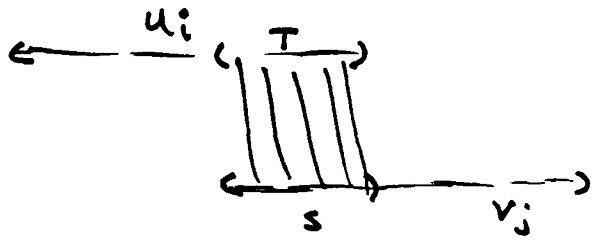
DELETE ALL EDGES BETWEEN IRREGULAR PAIRS.

DELETE EDGES BETWEEN PAIRS OF LOW DENSITY. (E.G. $< c/4$)

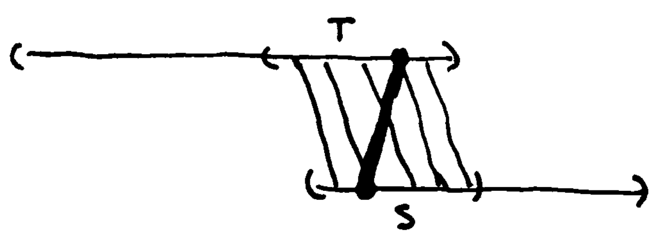
MANY EDGES REMAIN. $(\geq \frac{c}{2} n^2)$

SOME MATCHING M STILL HAS MANY EDGES $(\geq \frac{c}{2} n)$.

M MUST MEET SOME U_i AND SOME V_j IN LARGE SUBSETS T AND S, RESPECTIVELY. $(\geq \frac{1}{3} \frac{c}{2} \frac{n}{k})$



SINCE EDGES JOIN U_i AND V_j THIS IS A DENSE AND REGULAR PAIR. THEN $d(S,T)$ IS LARGE. EDGES FROM OTHER MATCHINGS MUST ALSO JOIN T AND S.



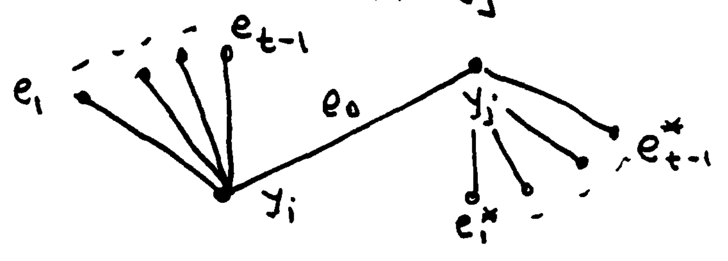
FOR THE PROOF THAT $\mathcal{A} \subseteq [n]^k$, $|\mathcal{A}| = n$, $k = cn$ CONTAINS A t-INTERSECTING SUBFAMILY OF SIZE $cn(1-o(1))$ WHEN $\underline{t > 2}$:

AGAIN CONSTRUCT THE G_ℓ, $\ell \in [n]$, NOW WITH $\{y_i, y_j\}$ AN EDGE IF $\ell \in A_i \cap A_j$ AND $|A_i \cap A_j| < t$.

CHOOSE MAXIMAL MATCHINGS M_ℓ AND IF $\{y_i, y_j\}$ IS IN ONE OF THEM DELETE $\{p, y_i\}$ AND $\{p, y_j\}$ FROM G FOR EACH $p \in A_i \cap A_j$.

SHOW THAT IF y_i AND y_j STILL HAVE A COMMON NEIGHBOR, THEN $|A_i \cap A_j| \geq t$.

SZEMERÉDI'S LEMMA CAN BE USED TO SHOW THAT WITH cn^2 EDGES IN $\leq n$ MATCHINGS THERE EXIST MATCHINGS $M_0, M_1, \ldots, M_{t-1}$ AND EDGES $e_0, e_1, \ldots, e_{t-1}$, $e_1^*, \ldots e_{t-1}^*$ WITH $e_0 = \{y_i, y_j\}$ AND FOR EACH μ, $1 \leq \mu \leq t-1$, $e_\mu, e_\mu^* \in M_\mu$, e_μ INCIDENT WITH y_i, e_μ^* INCIDENT WITH y_j.



IT IS NOT HARD TO SEE THAT THIS CONFIGURATION DOES NOT EXIST IN $\bigcup_{\ell=1}^{n} M_\ell$.

SO $\left|\bigcup_{\ell=1}^{n} M_\ell\right| = o(n^2)$.

THE ERDŐS-KO-RADO THEOREM FOR SMALL FAMILIES

R. DUKE, V. RÖDL

LET $[n] = \{1,2,\dots,n\}$ AND

$[n]^k = \{A : A \subseteq [n],\ |A| = k\}$ $(k < n/2)$

CALL $\mathcal{A}^t \subseteq [n]^k$ <u>t-INTERSECTING</u> IF

$|A_i \cap A_j| \geq t$ FOR EACH $A_i, A_j \in \mathcal{A}^t$.

HOW LARGE CAN SUCH A t-INTERSECTING FAMILY BE?

CHOOSING $\mathcal{A}^t = \{A : A \in [n]^k, \{1,2,\dots,t\} \subseteq A\}$ SHOWS THAT

$$\max |\mathcal{A}^t| \geq \binom{n-t}{k-t}.$$

<u>THEOREM</u> (ERDŐS-KO-RADO) GIVEN $k \geq t > 0$ THERE EXISTS $n_0 = n_0(k,t)$ SUCH THAT FOR $n \geq n_0$ IF $\mathcal{A}^t$ IS A t-INTERSECTING FAMILY, $\mathcal{A}^t \subseteq [n]^k$, THEN

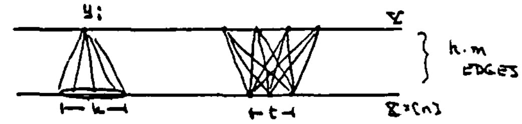
$$|\mathcal{A}^t| \leq \binom{n-t}{k-t}.$$

($n_0(k,t) = (k-t+1)(t+1)$, FRANKL, 1978; WILSON, 1984)

SINCE $\binom{n-t}{k-t} \sim \left(\frac{k}{n}\right)^t \binom{n}{k}$, IF $\mathcal{A} \subseteq [n]^k$, $|\mathcal{A}| = m \sim \binom{n}{k}$ AND $\mathcal{A}^t \subseteq \mathcal{A}$ IS t-INTERSECTING,

$$\max \frac{|\mathcal{A}^t|}{|\mathcal{A}|} \sim (k/n)^t.$$

LET $\mathcal{A} = \{A_i\}_{i=1}^m$ BE A SUBFAMILY OF $[n]^k$. CONSIDER THE BIPARTITE GRAPH G WITH VERTEX CLASSES $X = [n]$, $Y = \{y_1, y_2, \dots, y_m\}$ IN WHICH $\{\ell, y_i\}$ IS AN EDGE IFF $\ell \in A_i$.



SIMPLE AVERAGING SHOWS THAT THERE ARE t VERTICES IN X ALL JOINED TO THE SAME q VERTICES IN Y, WHERE

$$q \geq \binom{k}{t} \Big/ \binom{n}{t} \cdot m \sim (k/n)^t m.$$

SINCE THESE VERTICES OF Y CORRESPOND TO A t-INTERSECTING FAMILY

$$\max \frac{|\mathcal{A}^t|}{|\mathcal{A}|} \geq \frac{q}{m} \sim (k/n)^t.$$

LET $f_t(n,k,m)$ BE THE MINIMUM OF $\max \frac{|\mathcal{A}^t|}{|\mathcal{A}|}$ FOR $\mathcal{A}^t \subseteq \mathcal{A} \subseteq [n]^k$, $|\mathcal{A}| = m$, $\mathcal{A}^t$ t-INTERSECTING.

BY THE E-K-R THEOREM, FOR $m \sim \binom{n}{k}$ AND n LARGE, WE HAVE

$$f_t(n,k,m) \sim (k/n)^t.$$

FOR ALL m WE HAVE

$$f_t(n,k,m) \geq \frac{q}{m} \sim (k/n)^t.$$

HERE WE CONSIDER THIS FUNCTION FOR SMALL m.

IN PARTICULAR WE HAVE

<u>THEOREM</u> FOR <u>EACH</u> t, IF $m = n$ AND $k = cn$, $0 < c < 1/2$, THEN FOR n SUFFICIENTLY LARGE

$$f_t(n,k,m) \sim c = (k/n).$$

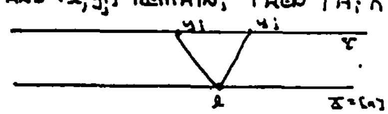
WE SHOW THAT FOR $\mathcal{A} \subseteq [n]^k$, $|\mathcal{A}| = n$, $k = cn$, THERE EXISTS A t-INTERSECTING FAMILY $\mathcal{A}^t \subseteq \mathcal{A}$ WITH <u>$|\mathcal{A}^t| = cn(1 - o(1))$</u>.

SKETCH OF THE PROOF FOR <u>$t = 2$</u>.

SUPPOSE $\mathcal{A} = \{A_i\}_{i=1}^n \subseteq [n]^k$, $k = cn$.

CONSIDER THE BIPARTITE GRAPH G AGAIN. G HAS cn^2 EDGES.

<u>CLAIM</u>. FOR n SUFFICIENTLY LARGE WE CAN DELETE $g(n)$ EDGES FROM G, $g(n) = o(n^2)$, SO THAT IF EDGES $\{\ell, y_i\}$ AND $\{\ell, y_j\}$ REMAIN, THEN $|A_i \cap A_j| \geq 2$.



IF SO, WE ARE DONE!

SINCE THEN SOME VERTEX $\ell \in X$ STILL HAS DEGREE $\geq \frac{1}{n}(cn^2 - g(n)) = cn(1 - o(1))$.

ITS NEIGHBORS CORRESPOND TO A 2-INTERSECTING FAMILY.

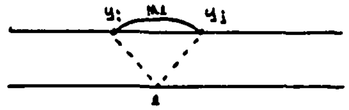
Proof of the claim:

For each $\ell \in [n]$ form a graph G_ℓ with vertex set $Y = \{y_1, y_2, \dots, y_n\}$ in which $\{y_i, y_j\}$ is an edge iff $A_i \cap A_j = \{\ell\}$.

In each G_ℓ choose a maximal matching, M_ℓ.

(Each edge of G_ℓ has an endpoint incident with an edge of M_ℓ.)

For each ℓ if $\{y_i, y_j\}$ is in M_ℓ delete the edges $\{\ell, y_i\}$ and $\{\ell, y_j\}$ from G.



If $\{\ell, y_i\}$ and $\{\ell, y_j\}$ remain in G, then $|A_i \cap A_j| \ge 2$.

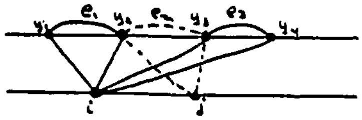
For otherwise, $\{y_i, y_j\}$ is in G_ℓ and at least one of $\{\ell, y_i\}, \{\ell, y_j\}$ would be missing.

It remains to show that $\left|\bigcup_{\ell=1}^{n} M_\ell\right| = o(n^2)$.

Note that in $\bigcup_{\ell=1}^{n} M_\ell$ we can not have

with $e_1, e_3 \in M_i$, $e_2 \in M_j$.

This would require



But then $i \in A_2 \cap A_3$, so $A_2 \cap A_3 \ne \{j\}$.

Theorem (Ruzsa, Szemerédi, 1978) For m sufficiently large if G is a bipartite graph with $2m$ vertices and cm^2 edges which are the union of $\le m$ matchings, then there exist matchings M_i and M_j and edges $e_1, e_3 \in M_i$, $e_2 \in M_j$, with e_2 incident with both e_1 and e_3.

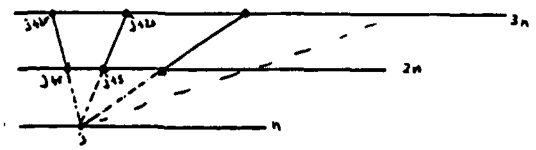
Since $\bigcup_{\ell=1}^{n} M_\ell$ does not have such edges, $\left|\bigcup_{\ell=1}^{n} M_\ell\right| = o(n^2)$.

This result follows from Szemerédi's Regularity Lemma and was used to show the following:

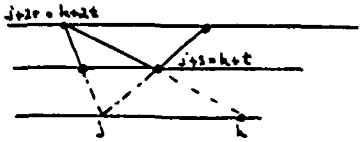
Let $v_3(n) = \max\{|S| : S \subseteq [n],\ S$ does not contain a 3-term arithmetic progression$\}$. Then $v_3(n) = o(n)$.

Suppose $S \subseteq [n]$, with $|S| = cn$, $0 < c < 1$.

For each $j \in [n]$ and each $r \in S$ join $j+r$ to $j+2r$. This yields a matching for each j, $1 \le j \le n$, in a bipartite graph with $n|S| = cn^2$ edges.



The Ruzsa, Szemerédi result insures that



$$\begin{array}{r} j+2r = k+2t \\ j+s = k+t \\ \hline 2r-s = t \end{array}$$

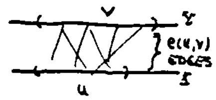
$$r = \frac{s+t}{2}$$

s, $r = \frac{s+t}{2}$, t form a 3-term arithmetic progression in S.

Szemerédi's Regularity Lemma (bipartite version)

For a bipartite graph H with vertex classes X and Y and $U \subseteq X$, $V \subseteq Y$:

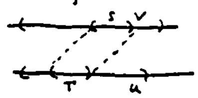
Density $d(U,V) = \dfrac{e(U,V)}{|U||V|}$



ε-regular pair (U,V)

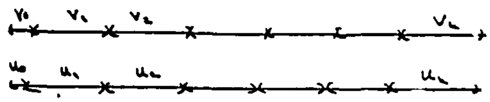
For each $T \subseteq U$, $S \subseteq V$ with $|T| \ge \varepsilon|U|$, $|S| \ge \varepsilon|V|$

$|d(T,S) - d(U,V)| < \varepsilon$



Suppose $|X| = |Y| = n$.

Theorem (Szemerédi) For each $\varepsilon > 0$, there exist integers $N(\varepsilon)$, $K(\varepsilon)$ such that for $n \ge N(\varepsilon)$ there are partitions $X = U_0 \cup U_1 \cup \dots \cup U_k$ and $Y = V_0 \cup V_1 \cup \dots \cup V_k$, $k \le K(\varepsilon)$, where $|U_0|, |V_0| < \varepsilon n$, $|U_1| = \dots = |U_k|$, $|V_1| = \dots = |V_k|$, and all but εk^2 of the pairs (U_i, V_j), $1 \le i, j \le k$, are ε-regular.



SZEMERÉDI'S REGULARITY LEMMA IMPLIES THE RESULT ON MATCHINGS (IN A BIPARTITE GRAPH).

SUPPOSE G IS A BIPARTITE GRAPH WITH n VERTICES IN EACH CLASS, cn^2 EDGES IN AT MOST n MATCHINGS.

APPLY THE REGULARITY LEMMA (WITH SUITABLE ε).

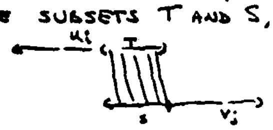
DELETE ALL EDGES BETWEEN IRREGULAR PAIRS.

DELETE EDGES BETWEEN PAIRS OF LOW DENSITY. $(\varepsilon_{ij} < \frac{c}{4})$

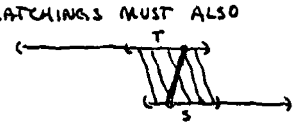
MANY EDGES REMAIN. $(\geq \frac{c}{2} n^2)$

SOME MATCHING M STILL HAS MANY EDGES $(\geq \frac{c}{2} n)$.

M MUST MEET SOME U_i AND SOME V_j IN LARGE SUBSETS T AND S, RESPECTIVELY. $(\geq \frac{1}{k} \frac{c}{2} \frac{n}{k})$



SINCE EDGES JOIN U_i AND V_j THIS IS A DENSE AND REGULAR PAIR. THEN $d(S,T)$ IS LARGE. EDGES FROM OTHER MATCHINGS MUST ALSO JOIN T AND S.



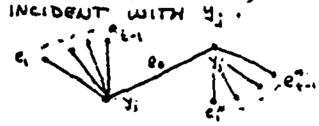
FOR THE PROOF THAT $a \subseteq [n]^k$, $|a| = n$, $k \leq cn$ CONTAINS A t-INTERSECTING SUBFAMILY OF SIZE $cn(1 - o(1))$ WHEN <u>$t > 2$</u>!

AGAIN CONSTRUCT THE G_ℓ, $\ell \in [n]$, NOW WITH $\{y_i, y_j\}$ AN EDGE IF $\ell \in A_i \cap A_j$ AND $|A_i \cap A_j| < t$.

CHOOSE MAXIMAL MATCHINGS M_ℓ AND IF $\{y_i, y_j\}$ IS IN ONE OF THEM DELETE $\{p, y_i\}$ AND $\{p, y_j\}$ FROM G FOR EACH $p \in A_i \cap A_j$.

SHOW THAT IF y_i AND y_j STILL HAVE A COMMON NEIGHBOR, THEN $|A_i \cap A_j| \geq t$.

SZEMERÉDI'S LEMMA CAN BE USED TO SHOW THAT WITH cn^2 EDGES IN $\leq n$ MATCHINGS THERE EXIST MATCHINGS $M_0, M_1, \dots, M_{t-1}$ AND EDGES $e_0, e_1, \dots, e_{t-1}$, $e_1^*, \dots e_{t-1}^*$ WITH $e_0 = \{y_i, y_j\}$ AND FOR EACH μ, $1 \leq \mu \leq t-1$, $e_\mu, e_\mu^* \in M_\mu$, e_μ INCIDENT WITH y_i, e_μ^* INCIDENT WITH y_j.



IT IS NOT HARD TO SEE THAT THIS CONFIGURATION DOES NOT EXIST IN $\bigcup_{\ell=1}^{n} M_\ell$.

SO $|\bigcup_{\ell=1}^{n} M_\ell| = o(n^2)$.

Critical m-neighbor-connected Graphs

Prof. Margaret B. Cozzens
Department of Mathematics
Northeastern University

APPLICATIONS AND BACKGROUND

Gunther and Hartnell in 1978 introduced the idea of neighbor connected graphs to model a spy network.
The vertices of a graph G are stations or people, the edges of G represent lines of communication.

If a station is destroyed, the adjacent stations will be betrayed so that the betrayed stations become useless to the network as a whole.

Therefore, we want to see what happens to a network when not only vertices are removed, but when neighborhoods of vertices are removed.

The ultimate goal is to design networks with high neighbor connectivity at least cost, so that the network communications are compromised the least in attack scenarios.

DEFINITIONS

Let G be a graph with υ vertices and ε edges.

closed neighborhood of u: $N[u] = \{u\} \cup N(u)$

a subverted vertex u: N[u] is deleted from G

G/S: G - N[S] where S is a set of vertices of G

S is a cut-strategy if G/S is empty, complete or disconnected

G is m-neighbor connected if
$m = \min\{|S|: S \text{ is a cut-strategy for } G\}$

K(G) denotes the neighbor connectivity of G

G is critically m-neighbor connected if $K(G) = m$, but $K(G/\{u\}) = m-1$ for all $u \in V(G)$

G is minimum critically m-neighbor connected if no critically m-neighbor connected graph with the same number of vertices has fewer edges than G

CONSTRUCTION OF NEW GRAPHS

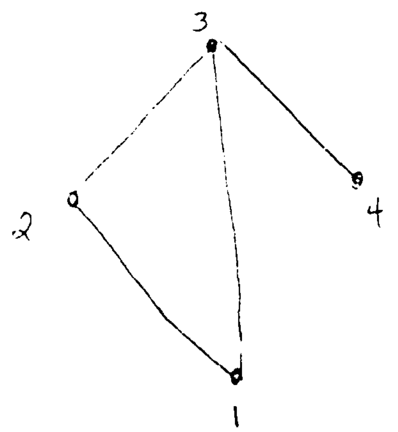
Given a graph G, create the collection $\mathscr{G}_G$:

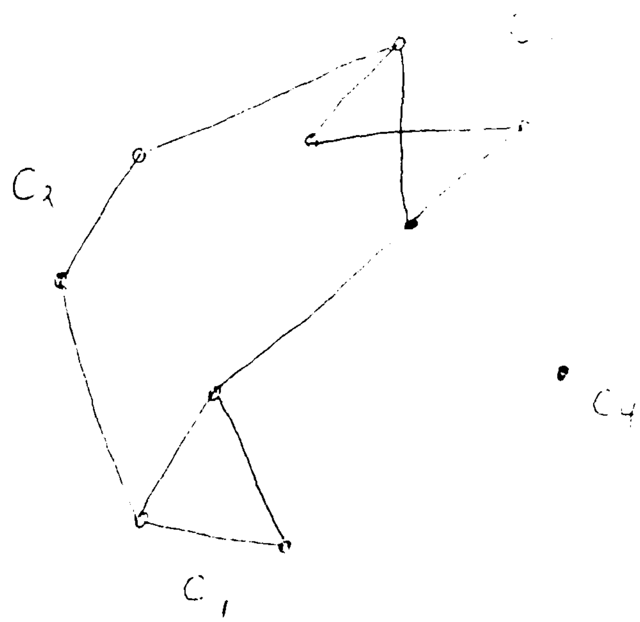
(i) Each vertex u of G is replaced by a clique C_u of order $\geq$ degree(u)

(ii) C_{u_1} and C_{u_2} are joined by one edge if and only if u_1 and u_2 are adjacent in G

(iii) Each vertex of C_u is adjacent to at most one vertex not in C_u

EXAMPLE





G

in $\mathscr{G}_G$

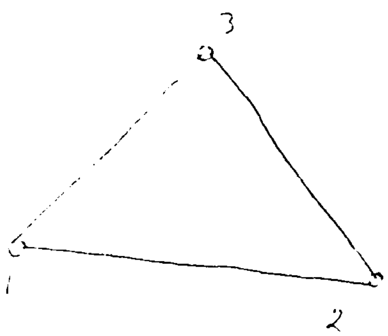
Given a graph G, create the collection $\mathcal{H}_G$ as follows:

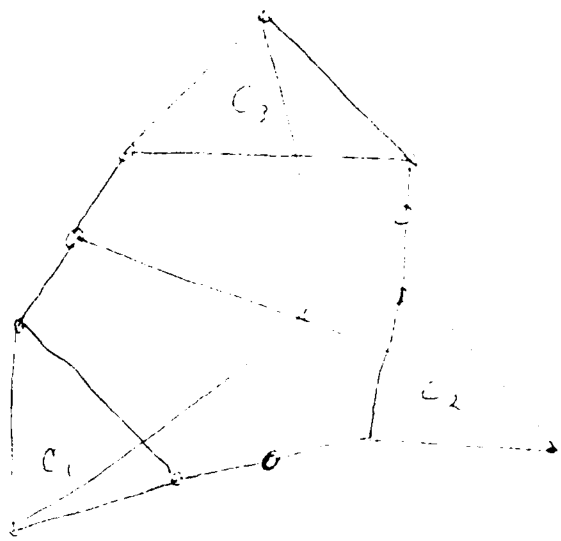
(i) Each vertex u of G is replaced by a clique of order $\geq$ degree(u)

(ii) Each clique is connected to another clique through a vertex called the courier if and only if the corresponding two vertices are connected in G.

(iii) Each vertex of a clique is connected to at most one courier.

EXAMPLE



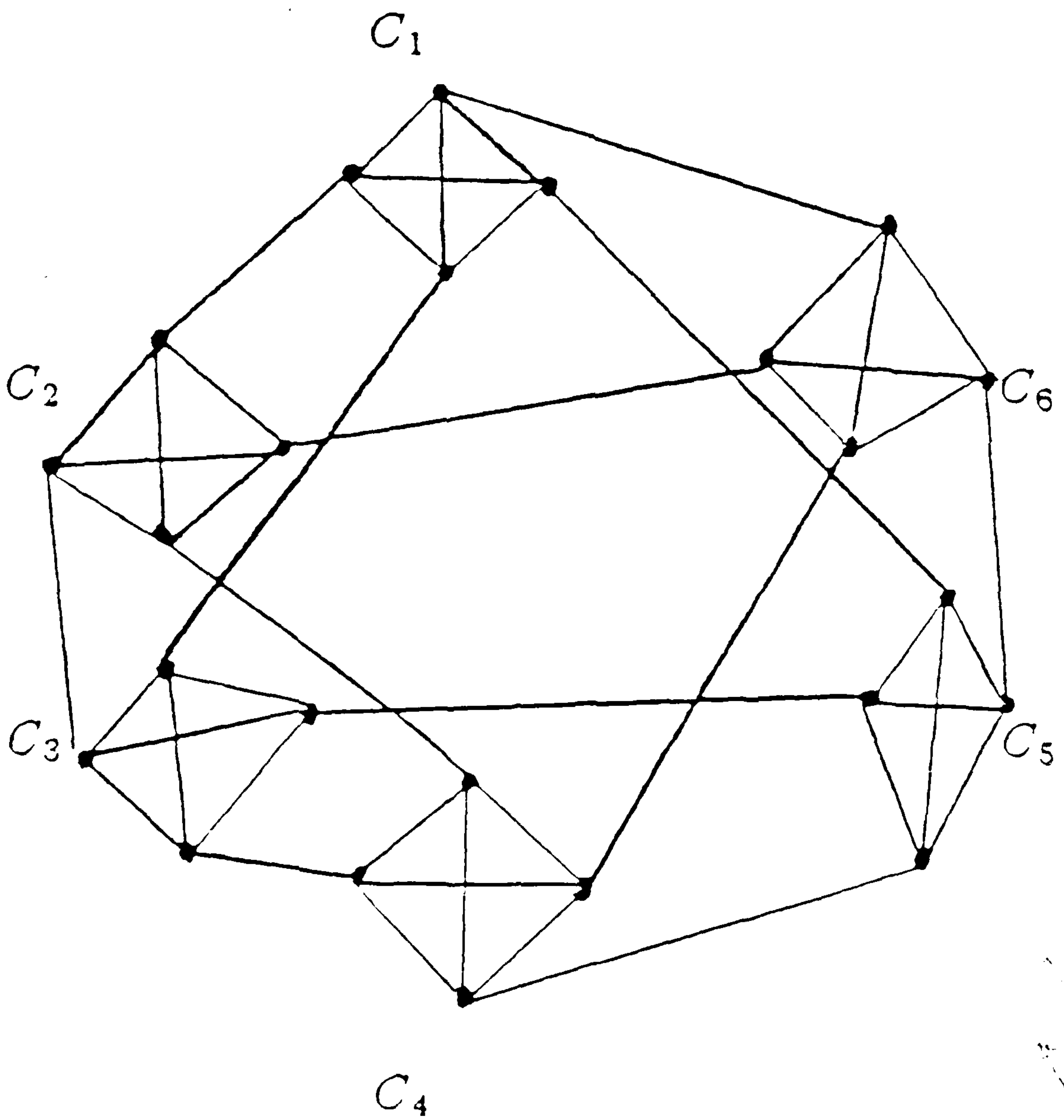


G

in $\mathcal{H}_G$

THEOREM 1: If G is an m-connected graph then each member of $\mathcal{G}$ is an m-neighbor connected graph.

THEOREM 2: For any positive integers m and n such that $m > 1$ and $n \geq m+1$, there is a class of *critically m-neighbor connected graphs*, each of which has n cliques.



$m = 4 \quad n = 6$ $\quad H_{4,6}$

THEOREM 3: Let m be a positive integer. If G is minimum critcally m-neighbor connected with order υ and ε edges then

$$\lceil 1/2 m\upsilon \rceil \leq \varepsilon \leq \lceil 1/2 m\upsilon + 1/2 mr \rceil$$

where r is the remainder of υ/m.

COROLLARY: If the order of G, υ, is a multiple of m and G is a minimum critically m-neighbor connected graph then $\varepsilon = \lceil 1/2 m\upsilon \rceil$.

RELATIONSHIP WITH OTHER PARAMETERS

The neighbor-connectivity number is less than or equal to the domination number.

$$K(G) \leq \beta(G)$$

Therefore:

1. If a connected graph G does not contain P_4 or C_4 as induced subgraphs then $K(G) = 1$.
2. If a connected graph G does not contain P_5 or C_5 or K_{3+p} as induced subgraphs then $K(G) \leq 2$.

The neighbor-connectivity number is less than or equal to the connectivity number.

$$K(G) \leq \kappa(G)$$

QUESTIONS:

1. When are they the same?
2. What graphs on υ vertices maximize both the connectivity and the neighbor connectivity simultaneously?

Define the **vertex-neighbor integrity** of a graph G to be:

$$NI(G) = \min \{|S| + w(G/S)\}$$

where w(G/S) is the size of the largest component in G/S and the minimum is taken over all cut strategies S.

3. For fixed υ, what graphs on υ vertices maximize the vertex-neighbor integrity?

4. For fixed υ, what graphs on υ vertices maximize the vertex-neighbor integrity and the neighbor connectivity simultaneously?

APPLICATIONS AND BACKGROUND

Gunther and Hartnell in 1978 introduced the idea of neighbor connected graphs to model a spy network.
The vertices of a graph G are stations or people, the edges of G represent lines of communication.

If a station is destroyed, the adjacent stations will be betrayed so that the betrayed stations become useless to the network as a whole.

Therefore, we want to see what happens to a network when not only vertices are removed, but when neighborhoods of vertices are removed.

The ultimate goal is to design networks with high neighbor connectivity at least cost, so that the network communications are compromised the least in attack scenarios.

DEFINITIONS

Let G be a graph with υ vertices and e edges.

closed neighborhood of u: $N[u] = \{u\} \cup N(u)$

a subverted vertex u: $N[u]$ is deleted from G

G/S: $G - N[S]$ where S is a set of vertices of G

S is a cut-strategy if G/S is empty, complete or disconnected

G is m-neighbor connected if
$m = \min\{|S|: S \text{ is a cut-strategy for } G\}$

K(G) denotes the neighbor connectivity of G

G is critically m-neighbor connected if $K(G) = m$, but $K(G/\{u\}) = m-1$ for all $u \in V(G)$

G is minimum critically m-neighbor connected if no critically m-neighbor connected graph with the same number of vertices has fewer edges than G

CONSTRUCTION OF NEW GRAPHS

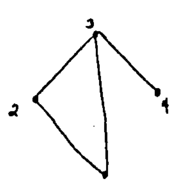
Given a graph G, create the collection $\mathscr{G}_G$:

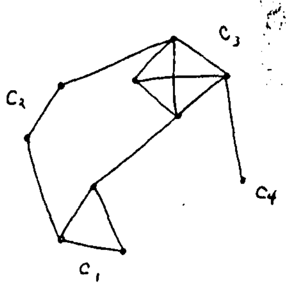
(i) Each vertex u of G is replaced by a clique C_u of order $\geq$ degree(u)

(ii) C_{u_1} and C_{u_2} are joined by one edge if and only if u_1 and u_2 are adjacent in G

(iii) Each vertex of C_u is adjacent to at most one vertex not in C_u

EXAMPLE





G in $\mathscr{G}_G$

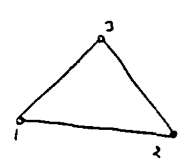
Given a graph G, create the collection $\mathscr{H}_G$ as follows:

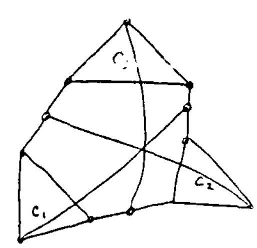
(i) Each vertex u of G is replaced by a clique of order $\geq$ degree(u)

(ii) Each clique is connected to another clique through a vertex called the courier if and only if the corresponding two vertices are connected in G.

(iii) Each vertex of a clique is connected to at most one courier.

EXAMPLE

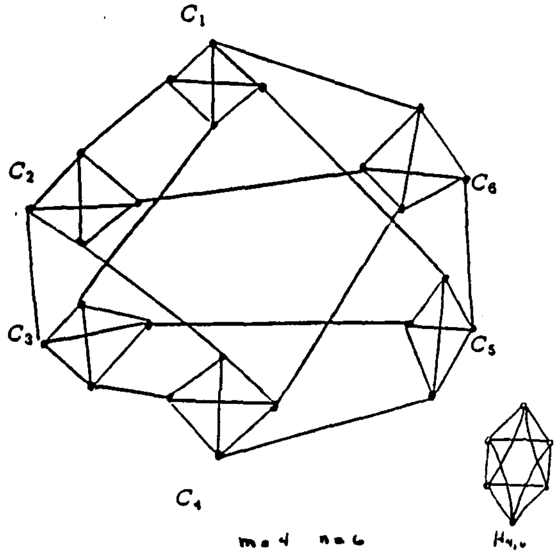




G in $\mathscr{H}_G$

THEOREM 1: If G is an m-connected graph then each member of $\mathscr{G}$ is an m-neighbor connected graph.

THEOREM 2: For any positive integers m and n such that $m > 1$ and $n \geq m+1$, there is a class of critically m-neighbor connected graphs, each of which has n cliques.



THEOREM 3: Let m be a positive integer. If G is minimum critcally m-neighbor connected with order υ and ε edges then

$$\lceil \tfrac{1}{2} m\upsilon \rceil \leq \varepsilon \leq \lceil \tfrac{1}{2} m\upsilon + \tfrac{1}{2} mr \rceil$$

where r is the remainder of υ/m.

COROLLARY: If the order of G, υ, is a multiple of m and G is a minimum critically m-neighbor connected graph then $\varepsilon = \lceil \tfrac{1}{2} m\upsilon \rceil$.

RELATIONSHIP WITH OTHER PARAMETERS

The neighbor-connectivity number is less than or equal to the domination number.

$$K(G) \leq \beta(G)$$

Therefore:

1. If a connected graph G does not contain P_4 or C_4 as induced subgraphs then $K(G) = 1$.
2. If a connected graph G does not contain P_5 or C_5 or K_{3+p} as induced subgraphs then $K(G) \leq 2$.

The neighbor-connectivity number is less than or equal to the connectivity number.

$$K(G) \leq \kappa(G)$$

QUESTIONS:

1. When are they the same?
2. What graphs on υ vertices maximize both the connectivity and the neighbor connectivity simultaneously?

Define the vertex-neighbor integrity of a graph G to be:

$$NI(G) = \min \{|S| + w(G/S)\}$$

where $w(G/S)$ is the size of the largest component in G/S and the minimum is taken over all cut strategies S.

3. For fixed υ, what graphs on υ vertices maximize the vertex-neighbor integrity?

4. For fixed υ, what graphs on υ vertices maximize the vertex-neighbor integrity and the neighbor connectivity simultaneously?

Cancellation and Consecutive Sets

Prof. Douglas R. Shier
Department of Mathematics
College of William and Mary

CANCELLATION AND CONSECUTIVE SETS

D. R. Shier

M. H. McIlwain*

* Supported by NSF-REU at College of William & Mary, Summer 1990.

COHERENT SYSTEMS

In general, have a set of components

$$E = \{1, 2, \dots, n\}$$

Component i: fails with probability $q_i = 1 - p_i$
(independently)

For subsets $X \subseteq E$

$$\Phi(X) = \begin{cases} 1 & \text{if system operates when components in } X \text{ operate, } E - X \text{ fail} \\ 0 & \text{otherwise} \end{cases}$$

Coherent system: $X \subseteq Y \Rightarrow \Phi(X) \leq \Phi(Y)$

pathset: minimal $S \subseteq E$ such that $\Phi(S) = 1$.

A coherent system completely described by E and $\mathcal{S}$, collection of pathsets.

<u>PROBLEM</u>: Calculate $R = Pr[\Phi(X) = 1]$.

If $E_i \sim$ all components in pathset S_i operate $\Rightarrow$

$$R = Pr[E_1 \cup E_2 \cup \cdots \cup E_k]$$

Inclusion-Exclusion Approach

Coherent system $(E, \mathcal{S})$

components $E = \{1, 2, \ldots, n\}$

pathsets $\mathcal{S} = \{S_1, S_2, \ldots, S_k\}$

WHEN is there interesting cancellation in I/E formula?

$$R = Pr[E_1 \cup E_2 \cup \cdots \cup E_k]$$
$$= \sum_i Pr[E_i] - \sum_{i<j} Pr[E_i E_j] + \cdots$$

Ex. 1: $S_1 = \{1, 2, 4\}$, $S_2 = \{2, 3\}$, $S_3 = \{1, 3, 4\}$

$$R = p_1 p_2 p_4 + p_2 p_3 + p_1 p_3 p_4 - 2\, p_1 p_2 p_3 p_4$$

Ex. 2: $S_1 = \{1, 2\}$, $S_2 = \{2, 3, 4\}$, $S_3 = \{3, 4, 5\}$, $S_4 = \{5, 6\}$

$$R = p_1 p_2 + p_2 p_3 p_4 + p_3 p_4 p_5 + p_5 p_6 - p_1 p_2 p_3 p_4$$
$$- p_1 p_2 p_5 p_6 - p_2 p_3 p_4 p_5 - p_3 p_4 p_5 p_6 + p_1 p_2 p_3 p_4 p_5 p_6$$

<u>9</u> terms (± 1) versus <u>15</u> possible.

CONSECUTIVE SYSTEMS

$\mathcal{S} = \{S_1, S_2, \ldots, S_k\}$ is a consecutive system on $E = \{1, 2, \ldots, n\}$ if each S_j contains consecutive elements of E: $S_j = [\ell_j, r_j]$.

<u>MAIN RESULT</u>. The ± 1 property holds for consecutive systems. Moreover, there is a nice interpretation of $0, +1, -1$ coeffs.

It suffices to study the coefficients

$$d(i, i+1, \ldots, n)$$

of $p_i p_{i+1} \cdots p_n$ in the I/E expansion for $\mathcal{S}$.

<u>FUNDAMENTAL TOOL</u>.

$$Pr[\Phi = 1] = (1 - p_e)\, Pr[\Phi = 1 \mid \underbrace{\bar{e}}_{\text{fails}}] + p_e\, Pr[\Phi = 1 \mid \underbrace{e}_{\text{works}}]$$

Can be repeatedly applied to find $d(i, i+1, \ldots, n)$.

EXAMPLE

$S_6 : \{1, 2, 3\}$
$S_5 : \{3, 4, 5, 6\}$
$S_4 : \{4, 5, 6, 7\}$ ← consecutive system
$S_3 : \{\cancel{6}, \cancel{7}, \cancel{8}\}$
$S_2 : \{\cancel{7}, \cancel{8}, 9\}$
$S_1 : \{9, 10, 11\}$

In $\{S_1\}$: $d(9, \ldots, 11) = +1$

In $\{S_1, S_2\}$: $d(7, \ldots, 11) = -1$

In $\{S_1, S_2, S_3\}$: $d(6, \ldots, 11) = ?$

$$Pr[\Phi = 1] = (1-p_6)\, Pr[\Phi = 1 \mid \bar{6}] + p_6\, Pr[\Phi = 1 \mid 6]$$

$$Pr[\Phi = 1 \mid 6] = (1-p_7)\, Pr[\Phi = 1 \mid 6\bar{7}] + p_7\, Pr[\Phi = 1 \mid 67]$$

$$Pr[\Phi = 1 \mid 67] = (1-p_8)\, Pr[\Phi = 1 \mid 67\bar{8}] + p_8\, \underbrace{Pr[\Phi = 1 \mid 678]}_{\equiv 1}$$

Now equate coeffs of $p_6 p_7 \cdots p_{11}$:

$$d(6, \ldots, 11) = -\{\underbrace{d(7, \ldots, 11 \mid \bar{6})}_{-1} + \underbrace{d(8, \ldots, 11 \mid 6\bar{7})}_{0} + \underbrace{d(9, \ldots, 11 \mid 67\bar{8})}_{+1}\}$$

$$= \underline{0}$$

RECURSION

If $S_j = [\ell_j, r_j]$ then

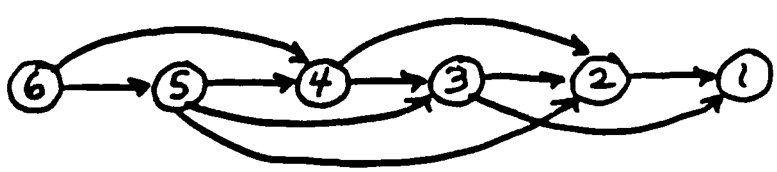
$$d(\ell_j,\dots,n) = -\left[\begin{array}{c} d(\ell_j+1,\dots,n \mid \bar{\ell}_j) + d(\ell_j+2,\dots,n \mid \ell_j\ \overline{\ell_j+1}) \\ + \dots + d(r_j+1,\dots,n \mid \ell_j,\dots,\bar{r}_j) \end{array}\right]$$

Certain of the terms are automatically 0; others are $d(r,\dots,n)$ in the subsystem $\{S_1,\dots,S_m\}$.

GIVEN sets $S_k, S_{k-1},\dots,S_1$ ordered by increasing ℓ_j, DEFINE consecutive union graph, with vertex v for each set S_v and directed edges (v,w), $v > w$, if $S_v \cup S_w$ is consecutive: $r_v + 1 \geq \ell_w$.

S_6 : $\{1,2,3\}$
S_5 : $\{3,4,5,6\}$
S_4 : $\{4,5,6,7\}$
S_3 : $\{6,7,8\}$
S_2 : $\{7,8,9\}$
S_1 : $\{9,10,11\}$

$x_i = d(\ell_i, \ell_{i+1},\dots,n)$



x_6	x_5	x_4	x_3	x_2	x_1
-1	0	1	0	-1	1

$$\boxed{x_i = -\sum_{r=1}^{d_i^+} x_{i-r}}$$

d_i^+ = outdegree of i

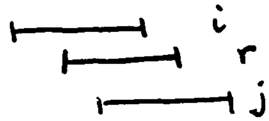
recursion on sets, not components.

Induction shows $x_i \in \{-1,0,1\}$.

C.U. GRAPHS

- Which graphs can arise as  graphs G?

NOTE: $(i,j) \in G \Rightarrow (i,r) \in G, \quad i < r \leq j$
$\qquad\qquad\qquad\quad \Rightarrow (r,j) \in G, \quad i \leq r < j$

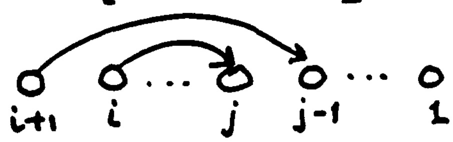


What conditions on d_i^+?

Always have $d_1^+ = 0$ and for $i > 1$

$$1 \leq d_i^+ \leq i-1$$

$$d_{i+1}^+ \leq d_i^+ + 1$$



These conditions on (consecutive) outdegrees characterize C.U. graphs; e.g.

$$\{d_4, d_3, d_2\} = \left.\begin{matrix} 1\ 1\ 1 \\ 2\ 1\ 1 \\ 1\ 2\ 1 \\ 2\ 2\ 1 \\ 3\ 2\ 1 \end{matrix}\right\} \text{ 5 such graphs for } k = 4$$

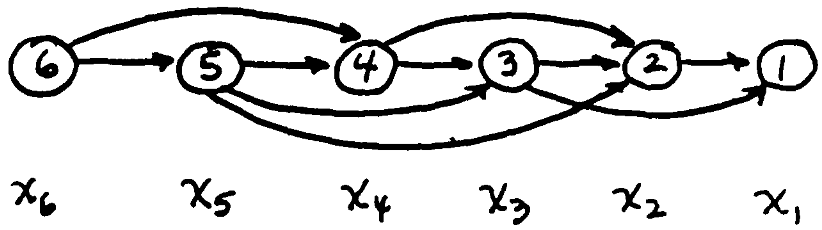
Can show

$$\#\text{ C.U. graphs on } k \text{ vertices} = C_{k-1} = \frac{1}{k}\binom{2k-2}{k-1}$$

ANOTHER VIEWPOINT

Characterize when $x_r = d(\ell_r, \ldots, n)$ is $0, -1, +1$?

Recall example:



The $\{x_r\}$ satisfy:

$$
\begin{aligned}
x_1 &= 1 \\
x_2 &= -x_1 \\
x_3 &= -x_2 - x_1 \\
x_4 &= -x_3 - x_2 \\
x_5 &= -x_4 - x_3 - x_2 \\
x_6 &= -x_5 - x_4
\end{aligned}
$$

or

$$
\boxed{
\begin{array}{llllllll}
x_1 & & & & & & = & 1 \\
x_1 & + x_2 & & & & & = & 0 \\
x_1 & + x_2 & + x_3 & & & & = & 0 \\
 & x_2 & + x_3 & + x_4 & & & = & 0 \\
 & x_2 & + x_3 & + x_4 & + x_5 & & = & 0 \\
 & & & x_4 & + x_5 & + x_6 & = & 0
\end{array}
}
$$

Solve $Ax = \varepsilon_1$, where $A = (a_{ij})$ is unit lower triangular

$a_{ij} = 1$ for $i - d_i^+ \le j \le i$

ε_1 is unit vector $(1, 0, \ldots, 0)^t$

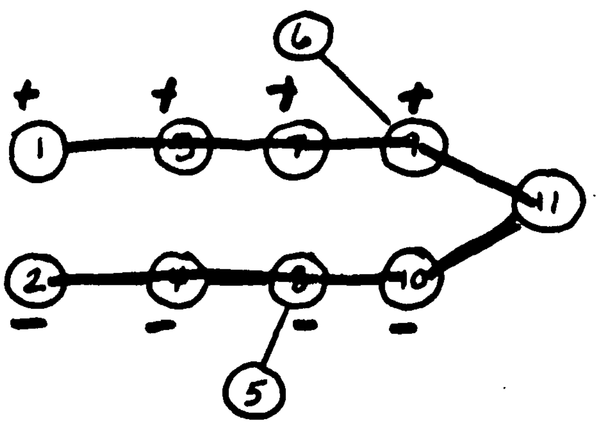
LINEAR SYSTEM

See that A has consecutive 1's in rows & in columns: particularly easy to solve $Ax = \varepsilon_1$.

LARGER EXAMPLE:

$$
\begin{array}{c|cccccccccc|c|}
 & + & - & + & - & & & + & - & + & - & \\
 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11} \\
\hline
s_1 & 1 & & & & & & & & & & \\
s_2 & 1 & 1 & & & & & & & & & \\
s_3 & & 1 & 1 & & & & & & & & \\
s_4 & & & 1 & 1 & & & & & & & \\
s_5 & & & 1 & 1 & 1 & & & & & & \\
s_6 & & & 1 & 1 & 1 & 1 & & & & & \\
s_7 & & & & 1 & 1 & 1 & 1 & & & & \\
s_8 & & & & & & 1 & 1 & 1 & & & \\
s_9 & & & & & & & & 1 & 1 & & \\
s_{10} & & & & & & & & & 1 & 1 & \\
\hline
s_{11} & & & & & & & & & & & 1 \\
\hline
\end{array}
=
\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$\Rightarrow x = (1, -1, 1, -1, 0, 0, 1, -1, 1, -1)$

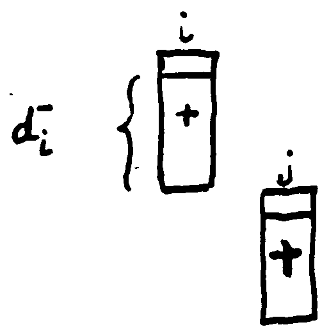


a path $1 \to 2$ using edge $(10,11)$

Since in general A is totally unimodular, the system $Ax = e_1$ has solution x with entries $\in \{-1, 0, 1\}$.

<u>NOTE</u>: $+1$ entries produce $(1, 1, \ldots, 1, 0, \ldots, 0)^t$
-1 " " $(0, -1, \ldots, -1, 0, \ldots, 0)^t$

Define another graph $T(A)$ to indicate how these positive (negative) columns fit together



vertex $i \sim$ column i
edge (i, j) for $j = i + d_i^- + 1$

Convenient to append a new row & column to A with $a_{k+1,k+1} = 1$, other entries 0.

Then each $i \neq k+1$ has a unique successor $j \Rightarrow$ $T(A)$ is a tree rooted at vertex $k+1$.

RESULT

<u>Theorem</u>. Let P be the path joining 1 and 2 in $T(A)$.

$$P \text{ contains } (k, k+1) \Leftrightarrow x_k \neq 0.$$

Moreover, in this case, $x_k = (-1)^{|P|+1}$

Comments:

1. $T(A) = T(\mathcal{S})$ can be directly constructed from $\mathcal{S} = \{S_k, \ldots, S_1\}$: $(i,j) \in T(A) \Leftrightarrow j = i + d_i^- + 1$.

2. Once $T(A)$ is constructed, the path joining j and $j+1$ determines the coefficient $d(\ell_k, \ldots, r_j)$ in
$$S_k = [\ell_k, r_k]$$
$$\vdots$$
$$S_j = [\ell_j, r_j]$$

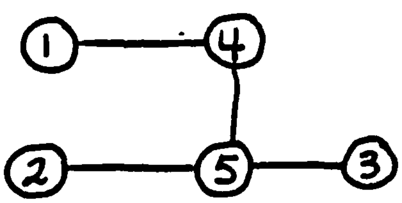
3. By coalescing vertices $k, k+1 \to k$, get the appropriate tree for system with S_k removed.

EXAMPLE

S_4 : $\{1, 2\}$
S_3 : $\{2, 3, 4\}$
S_2 : $\{3, 4, 5\}$
S_1 : $\{5, 6\}$

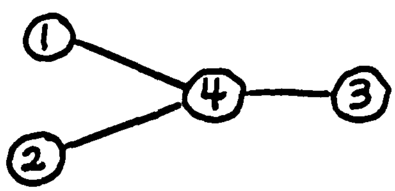
Nonzero Coeffs.

$\{S_4, \ldots, S_i\}$



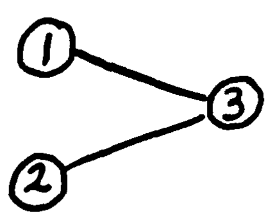
1-2: $+\ p_1 p_2 p_3 p_4 p_5 p_6$
3-4: $-\ p_1 p_2 p_3 p_4$
4-5: $+\ p_1 p_2$

$\{S_3, \ldots, S_i\}$



2-3: $-\ p_2 p_3 p_4 p_5$
3-4: $+\ p_2 p_3 p_4$

$\{S_2, \ldots, S_i\}$



1-2 : $-\ p_3 p_4 p_5 p_6$
2-3 : $+\ p_3 p_4 p_5$

$\{S_1, \ldots, S_i\}$

1-2: $+\ p_5 p_6$

Nonconsecutive Terms

Construction of $T(\mathcal{S})$ enables determination of coefficient $d(v, v+1, \ldots, w)$ for $p_v p_{v+1} \cdots p_w$ in the I/E expansion of $\mathcal{S}$.

There can be other terms $A_1, A_2, \ldots, A_r$ each corresponding to (maximal) sets of consecutive elements: e.g. $A_1 = \{1, 2, 3\}$, $A_2 = \{5, 6, 7, 8\}$

<u>Theorem</u>. $d(A_1 A_2 \cdots A_r) = (-1)^{r+1} d(A_1)\, d(A_2) \cdots d(A_r)$

Previous example:

$$d(1,2,5,6) = -d(1,2)\, d(5,6) = -1$$

SUMMARY

Inclusion-exclusion expansion

$\Pr[E_1 \cup \dots \cup E_k]$

predictable cancellation?

S&P (1978)
K&P (1988)

Consecutive sets $S_1, S_2, \dots, S_k$

Recursion

consecutive union graph

uses outdegrees

Linear system

based on indegrees

$T(\delta)$

character of $j, j+1$ path in $T(\delta)$

Extension

column consecutive systems

$S_3: \{1,2,3\}$
$S_2: \{2,3,4,5\}$
$S_1: \{3,4,6\}$

Cancellation and Consecutive Sets

D. R. Shier

M. H. McIlwain*

* Supported by NSF-REU at College of William & Mary, Summer 1990.

Coherent Systems

In general, have a set of components

$E = \{1, 2, \dots, n\}$

Component i: fails with probability $q_i = 1 - p_i$ (independently)

For subsets $X \subseteq E$

$$\Phi(X) = \begin{cases} 1 & \text{if system operates when components in } X \text{ operate, } E - X \text{ fail} \\ 0 & \text{otherwise} \end{cases}$$

Coherent system: $X \subseteq Y \Rightarrow \Phi(X) \le \Phi(Y)$

pathset: minimal $S \subseteq E$ such that $\Phi(S) = 1$

A coherent system completely described by E and $\mathcal{S}$, collection of pathsets.

<u>PROBLEM</u>: Calculate $R = Pr[\Phi(X) = 1]$.

If E_i ~ all components in pathset S_i operate $\Rightarrow$

$$R = Pr[E_1 \cup E_2 \cup \dots \cup E_k]$$

Inclusion-Exclusion Approach

Coherent system $(E, \mathcal{S})$

components $E = \{1, 2, \dots, n\}$

pathsets $\mathcal{S} = \{S_1, S_2, \dots, S_k\}$

WHEN is there interesting cancellation in I/E formula?

$$R = Pr[E_1 \cup E_2 \cup \dots \cup E_k]$$
$$= \sum_i Pr[E_i] - \sum_{i<j} Pr[E_i E_j] + \dots$$

Ex. 1: $S_1 = \{1,2,4\}$, $S_2 = \{2,3\}$, $S_3 = \{1,3,4\}$

$$R = p_1p_2p_4 + p_2p_3 + p_1p_3p_4 - 2p_1p_2p_3p_4$$

Ex. 2: $S_1 = \{1,2\}$, $S_2 = \{2,3,4\}$, $S_3 = \{3,4,5\}$, $S_4 = \{5,6\}$

$$R = p_1p_2 + p_2p_3p_4 + p_3p_4p_5 + p_5p_6 - p_1p_2p_3p_4 - p_1p_2p_5p_6 - p_2p_3p_4p_5 - p_3p_4p_5p_6 + p_1p_2p_3p_4p_5p_6$$

<u>9</u> terms (± 1) versus <u>15</u> possible.

Consecutive Systems

$\mathcal{S} = \{S_1, S_2, \dots, S_k\}$ is a consecutive system on $E = \{1, 2, \dots, n\}$ if each S_j contains consecutive elements of E: $S_j = [\ell_j, r_j]$

<u>MAIN RESULT</u>. The ± 1 property holds for consecutive systems. Moreover, there is a nice interpretation of $0, +1, -1$ coeffs.

It suffices to study the coefficients

$$d(i, i+1, \dots, n)$$

of $p_i p_{i+1} \cdots p_n$ in the I/E expansion for $\mathcal{S}$.

<u>FUNDAMENTAL TOOL</u>.

$$Pr[\Phi = 1] = (1 - p_e)\, Pr[\Phi = 1 \mid \underset{\text{fails}}{\bar{e}}] + p_e\, Pr[\Phi = 1 \mid \underset{\text{works}}{e}]$$

Can be repeatedly applied to find $d(i, i+1, \dots, n)$.

EXAMPLE

$S_6: \{1,2,3\}$
$S_5: \{3,4,5,6\}$
$S_4: \{4,5,6,7\}$ ← consecutive system
$S_3: \{\cancel{6},\cancel{7},\cancel{8}\}$
$S_2: \{\cancel{7},\cancel{8},9\}$
$S_1: \{9,10,11\}$

In $\{S_1\}$: $d(9,\dots,11) = +1$

In $\{S_1,S_2\}$: $d(7,\dots,11) = -1$

In $\{S_1,S_2,S_3\}$: $d(6,\dots,11) = ?$

$$Pr[\Phi=1] = (1-p_6)\, Pr[\Phi=1|\bar{6}] + p_6\, \underline{Pr[\Phi=1|6]}$$

$$(1-p_7)\, Pr[\Phi=1|6\bar{7}] + p_7\, \underline{Pr[\Phi=1|67]}$$

$$(1-p_8)\, Pr[\Phi=1|67\bar{8}] + p_8\, \underbrace{Pr[\Phi=1|678]}_{\equiv 1}$$

Now equate coeffs of $p_6 p_7 \cdots p_{11}$:

$$d(6,\dots,11) = -\{\underbrace{d(7,\dots,11|\bar{6})}_{-1} + \underbrace{d(8,\dots,11|6\bar{7})}_{0} + \underbrace{d(9,\dots,11|67\bar{8})}_{+1}\}$$

$$= \underline{0}$$

RECURSION

If $S_j = [\ell_j, r_j]$ then

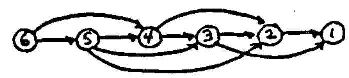
$$d(\ell_j,\dots,n) = -\left[\begin{array}{l} d(\ell_j+1,\dots,n \mid \bar{\ell}_j) + d(\ell_j+2,\dots,n \mid \ell_j\, \overline{\ell_j+1}) \\ + \dots + d(r_j+1,\dots,n \mid \ell_j,\dots,\bar{r}_j) \end{array}\right]$$

Certain of the terms† are automatically 0; others are $d(r,\dots,n)$ in the subsystem $\{S_1,\dots,S_m\}$.

GIVEN sets $S_k, S_{k-1},\dots,S_1$ ordered by increasing ℓ_j, DEFINE consecutive union graph, with vertex v for each set S_v and directed edges (v,w), $v>w$, if $S_v \cup S_w$ is consecutive: $r_v + 1 \geq \ell_w$.

$S_6: \{1,2,3\}$
$S_5: \{3,4,5,6\}$
$S_4: \{4,5,6,7\}$
$S_3: \{6,7,8\}$
$S_2: \{7,8,9\}$
$S_1: \{9,10,11\}$

$x_i = d(\ell_i, \ell_{i+1},\dots,n)$



x_6	x_5	x_4	x_3	x_2	x_1
-1	0	1	0	-1	1

$$\boxed{x_i = -\sum_{r=1}^{d_i^+} x_{i-r}}$$

d_i^+ = outdegree of i

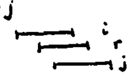
recursion on sets, not components.

Induction shows $x_i \in \{-1,0,1\}$.

C.U. GRAPHS

• Which graphs can arise as [illegible] graphs G?

NOTE: $(i,j)\in G \Rightarrow (i,r)\in G,\ i<r\leq j$
$\Rightarrow (r,j)\in G,\ i\leq r<j$

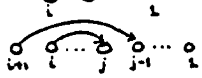


What conditions on d_i^+?

Always have $d_1^+ = 0$ and for $i>1$

$$1 \leq d_i^+ \leq i-1$$

$$d_{i+1}^+ \leq d_i^+ + 1$$



These conditions on (consecutive) outdegrees characterize C.U. graphs; e.g.

$\{d_4, d_3, d_2\} =$ 111, 211, 121, 221, 321 } 5 such graphs for $k=4$

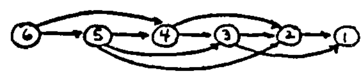
Can show

$$\#\ \text{C.U. graphs on } k \text{ vertices} = C_{k-1} = \frac{1}{k}\binom{2k-2}{k-1}$$

ANOTHER VIEWPOINT

Characterize when $x_r = d(\ell_r,\dots,n)$ is $0,-1,+1$?

Recall example:



$x_6 \quad x_5 \quad x_4 \quad x_3 \quad x_2 \quad x_1$

The $\{x_r\}$ satisfy:

$$\begin{aligned} x_1 &= 1 \\ x_2 &= -x_1 \\ x_3 &= -x_2 - x_1 \\ x_4 &= -x_3 - x_2 \\ x_5 &= -x_4 - x_3 - x_2 \\ x_6 &= -x_5 - x_4 \end{aligned}$$

or

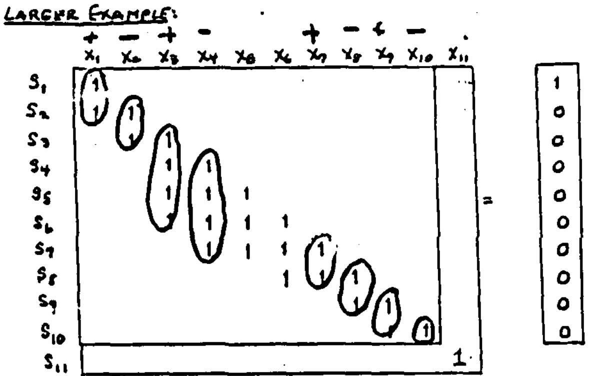
$$\boxed{\begin{array}{lr} x_1 & = 1 \\ x_1 + x_2 & = 0 \\ x_1 + x_2 + x_3 & = 0 \\ x_2 + x_3 + x_4 & = 0 \\ x_2 + x_3 + x_4 + x_5 & = 0 \\ x_4 + x_5 + x_6 & = 0 \end{array}}$$

Solve $Ax = e_1$, where $A = (a_{ij})$ is unit lower triangular
$a_{ij} = 1$ for $i - d_i^+ \leq j \leq i$
e_1 is unit vector $(1,0,\dots,0)^t$

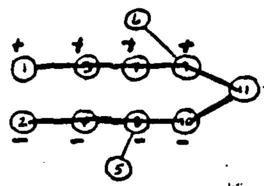
LINEAR SYSTEM

See that A has consecutive 1's in rows & in columns: particularly easy to solve $Ax = e_1$.

LARGER EXAMPLE:



$\Rightarrow x = (1,-1,1,-1,0,0,1,-1,1,-1)$



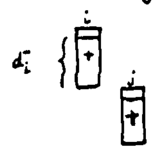
a path $1 \to 2$ using edge $(10,11)$

Since in general A is totally unimodular, the system $Ax = e_1$ has solution x with entries $\in \{-1,0,1\}$.

NOTE: +1 entries produce $(1,1,\dots,1,0,\dots,0)^t$

-1 " " $(0,-1,\dots,-1,0,\dots,0)^t$

Define another graph T(A) to indicate how these positive (negative) columns fit together



vertex i ~ column i

edge (i,j) for $j = i + d_i^- + 1$

Convenient to append a new row & column to A with $a_{k+1,k+1} = 1$, other entries 0.

Then each $i \neq k+1$ has a unique successor $j \Rightarrow$ T(A) is a tree rooted at vertex $k+1$.

RESULT

Theorem. Let P be the path joining 1 and 2 in T(A).

P contains $(k,k+1) \Leftrightarrow x_k \neq 0$.

Moreover, in this case, $x_k = (-1)^{|P|+1}$

Comments:

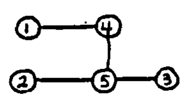
1. $T(A) = T(\mathcal{S})$ can be directly constructed from $\mathcal{S} = \{S_k, \dots, S_1\}$: $(i,j) \in T(A) \Leftrightarrow j = i + d_i^- + 1$.
2. Once T(A) is constructed, the path joining j and $j+1$ determines the coefficient $d(\ell_k, \dots, r_j)$ in
$$S_k = [\ell_k, r_k]$$
$$\vdots$$
$$S_j = [\ell_j, r_j]$$
3. By coalescing vertices $k, k+1 \to k$, get the appropriate tree for system with S_k removed.

EXAMPLE

S_4: $\{1,2\}$
S_3: $\{2,3,4\}$
S_2: $\{3,4,5\}$
S_1: $\{5,6\}$

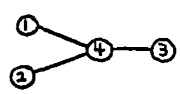
Nonzero Coeffs.

$\{S_4, \dots, S_i\}$



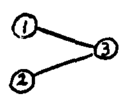
1-2: $+\ p_1 p_2 p_3 p_4 p_5 p_6$
3-4: $-\ p_1 p_2 p_3 p_4$
4-5: $+\ p_1 p_2$

$\{S_3, \dots, S_i\}$



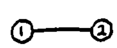
2-3: $-\ p_2 p_3 p_4 p_5$
3-4: $+\ p_2 p_3 p_4$

$\{S_2, \dots, S_i\}$



1-2: $-\ p_3 p_4 p_5 p_6$
2-3: $+\ p_3 p\ p_5$

$\{S_1, \dots, S_i\}$



1-2: $+\ p_5 p_6$

Nonconsecutive Terms

Construction of $T(\mathcal{S})$ enables determination of coefficient $d(v, v+1, \ldots, w)$ for $p_v p_{v+1} \cdots p_w$ in the I/E expansion of $\mathcal{S}$.

There can be other terms $A_1, A_2, \ldots, A_r$ each corresponding to (maximal) sets of consecutive elements: e.g. $A_1 = \{1,2,3\}$, $A_2 = \{5,6,7,8\}$

<u>Theorem</u>. $d(A_1 A_2 \cdots A_r) = (-1)^{r+1} d(A_1)\, d(A_2) \cdots d(A_r)$

Previous example:

$$d(1,2,5,6) = -d(1,2)\, d(5,6) = -1$$

Summary

Inclusion-exclusion expansion

- $Pr[E_1 \cup \ldots \cup E_k]$ — S & P (1978)
- predictable cancellation? — K & P (1987)

Consecutive sets $S_1, S_2, \ldots, S_k$

Recursion

- consecutive union graph
- uses outdegrees

Linear system

- based on indegrees
- $T(\mathcal{S})$
- character of $j, j+1$ path in $T(\mathcal{S})$

Extension

- column consecutive systems

S_3 : $\{1,2,3\}$
S_2 : $\{2,3,4,5\}$
S_1 : $\{3,4,6\}$

Andrew Sobczyk Memorial Lecture

The Local Ramsey Number and Local Colorings

Prof. Richard H. Schelp
Department of Mathematical Sciences
Memphis State University

The Local Ramsey Number and Local Colorings

by

R. H. Schelp

References

1) A. Gyárfás, J. Lehel, Zs. Tuza, R.H. Schelp, Ramsey Numbers for Local Colorings, Graphs and Comb. 3 (1987) 267–277.

2) A. Gyárfás, J. Lehel, J. Nešetřil, V. Rödl, R.H. Schelp, Zs. Tuza, Local k-Colorings of Graphs and Hypergraphs, JCTB 43(2) (1987) 127–139.

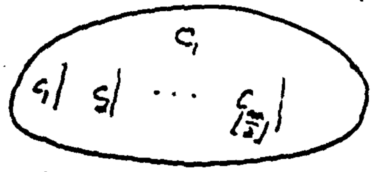
3) M. Truszczyński, Generalized Local Colourings of Graphs, to appear in JCTB.

4) R.A. Clapsadle and R.H. Schelp, Local Edge Colourings that are Global, in preparation.

Def. A *local k-coloring* of a graph H is a coloring of its edges in such a way that the edges incident to each vertex of H are colored by at most k different colors. The *local Ramsey number* $r^k_{loc}(G)$ of a graph G is the smallest positive integer m such that K_m contains a monochromatic copy of G under each local k-coloring.

Observations:

1) For k fixed and m large a local k-colouring of K_m can use a large number of colors independent of k.

$K_m =$ 

Here all edges of K_m are colored with color c_1 except for the shown $\lfloor \frac{m}{2} \rfloor$ matching edges.

2) If $r^k(G)$ denotes the usual Ramsey number, then $r^k(G) \le r^k_{loc}(G)$ since each coloring of a K_m by at most k colors is a local k-coloring.

Theorem 1 (Existence) For each $n \geq 3$, $k \geq 2$ $R^{k}_{loc}(G) \leq \left\lceil k^{k(n-2)+1} / (k-1) \right\rceil$

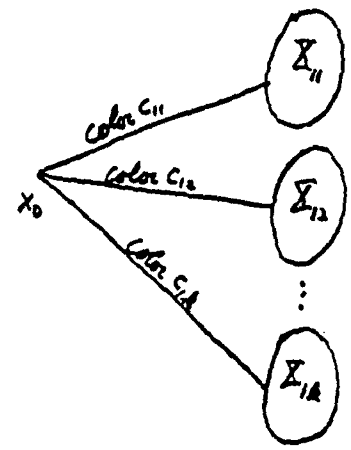
Proof. Let G be a local k-colored complete graph with $\geq \left\lceil k^{k(n-2)+1} / (k-1) \right\rceil$ vertices. We

show G contains a spanning tree T, rooted at a fixed vertex x_0, such that the following holds.

(i) Each $x \in V(T)$ has at most k successors $x_1, x_2, \cdots, x_s$ $(s \leq k)$ with different colors assigned to each edge xx_i, $1 \leq i \leq s$.

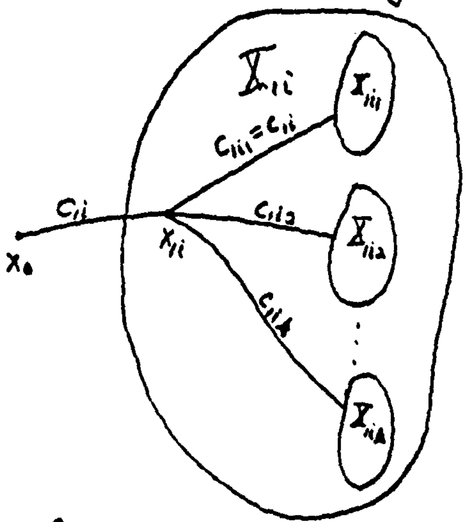
(ii) Edges xy and xz have the same color for $x, y, z \in V(T)$ when $x < y < z$, where the ordering corresponds to a partial order with x_0 as minimum element.

How T is found



A partition of $V(G)$.

Select a fixed vertex x_{1i} in each X_{1i} and partition X_{1i} as was just done for $V(G)$ replacing x_0 by x_{1i}.



Continue this process until a spanning tree is obtained.

By condition (i) there exists a path P from the root x_0 to a vertex y with at least $k(n-2)+1$ edges.

P: $x_0 \overset{c_{i_1}}{—} x_1 \overset{c_{i_2}}{—} x_2 \cdots x_{\ell-1} \overset{c_{i_\ell}}{—} x_\ell \cdots y$

By condition (ii) x_ℓ is incident to edges colored $c_{i_1}, c_{i_2}, \cdots, c_{i_\ell}$. Therefore the edges of P get at most k colors. Hence there exist $n-1$ edges, say $x_1y_1, x_2y_2, \cdots, x_{n-1}y_{n-1}$ with the same color. $x_1 \overset{c}{—} y_1$, $x_2 \overset{c}{—} y_2$ $\cdots$ $x_{n-1} \overset{c}{—} y_{n-1}$

Then by (ii) $\{x_1, x_2, \cdots, x_{n-1}, y_{n-1}\}$ spans a K_n in color c.

Theorem 2

(i) $r^2_{loc}(K_n) = r^2(K_n)$ for all n.

(ii) $r^2_{loc}(K_{n-m}+\overline{K}_m) = r^2(K_{n-m}+\overline{K}_m)$, $(n,m)\neq(3,2)$, $n\geq 2m-1$.

(iii) $r^2_{loc}(C_n) = r^2(C_n)$ for all $n\geq 3$.

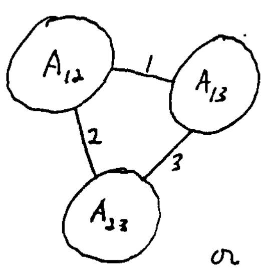
(iv) $r^2_{loc}(P_{2m}) = r^2(P_{2m}) = 3m-1$, $m\geq 1$

(v) $r^2_{loc}(P_{2m+1}) = r^2(P_{2m+1})+1 = 3m+1$, $m\geq 1$

(vi) $r^2_{loc}(nK_3) = 7n-2$, $n\geq 2$,
$r^2(nK_3) = 5n$, $n\geq 2$.

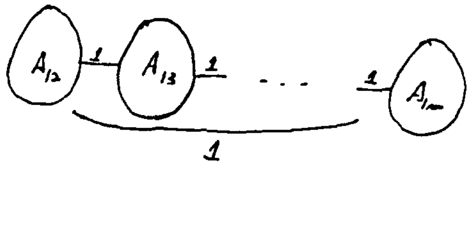
(vii) For a connected graph G
$r^2_{loc}(G)\geq 3|V(G)|/2$ and there are trees T such that $r^2(T)\leq \lfloor 4/3\,|V(T)|-1\rfloor$

Reason local 2-coloring problem is easier than for arbitrary k: Any local 2-coloring of K_n looks like



All edges in A_{ij} colored with either color i or color j.

or



Theorem 3.

(i) $r^k_{loc}(S_n) = k(n-1)+2$, $k,n\geq 1$, where S_n is star on n edges.

(ii) $r^k_{loc}(P_4) = \begin{cases} 2k+2 & \text{if } k\equiv 0,1 \pmod 3 \\ 2k+1 & \text{if } k\equiv 2 \pmod 3 \end{cases}$

(iii) $r^3_{loc}(K_3) = 17$

(iv) (Truszczynski-Tuza)
For G a connected graph
$r^k_{loc}(G)/r^k(G)\leq c_k$ (c_k - constant depending on k)

(v) For $n\geq 1$, $t\geq 2$
$r^2_{loc}(nK_t)\geq n(t^2-t+1)-t+1$ and for n large WRT t
$r^2(nK_t)\leq (2t-1)n+c_t$

Nature of local k-colorings

Theorem 4. If G is a locally k-colored graph, then for some monochromatic subgraph G_i, the average degree $d^*(G_i)\geq d^*(G)/k$.

Corollary. If G is locally k-colored, then it contains a monochromatic subgraph of minimum degree $\geq d^*(G)/2k$.

It is easy to see that if the edges of G are colored by at most k colors and $\chi(G)\geq m^k+1$, then G contains a monochromatic subgraph G' with $\chi(G')\geq m+1$.

This no longer holds for k-colorings.

Theorem 5. There exist graphs with arbitrary large chromatic number with local 2-colorings such that each monochromatic graph is bipartite.

Def. Let K_n^r denote the complete r-uniform hypergraph. A local k-coloring of K_n^r is a coloring of its edges such that the set of edges containing any $(r-1)$-element subset of vertices are colored by at most k different colors.

Theorem 6. (Existence - Ramsey Number for Hypergraphs)

Let k, r, and n be positive integers, $r \le n$. Then there exists an $N = N(k,r,n)$ such that every local k-coloring of K_N^r contains a monochromatic K_n^r.

As an application of Theorem 6 can prove the following theorem.

Theorem 7. For all bipartite graphs B and for all k there exists a bipartite graph B' such that when B' is locally k-colored, then it contains a monochromatic copy of B as an induced subgraph of B'.

Theorem 8. Let G be a graph on n vertices with $\Delta(G) \le d$. Then for each k there exists a function $c = c(k,d)$ such that $r_{loc}^k(G) \le cn$.

A Generalization by M. Truszczyński

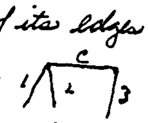
Def Let k be a fixed positive integer and let H be a fixed graph with at least $k+1$ edges. We say a graph G has been given a local (H,k)-Coloring (or simply an (H,k)-coloring) if each subgraph of G isomorphic to H has its edges colored by at most k different colors.

Note that a local k-coloring of K_n is a $(K_{1,n-1}, k)$-coloring.

The (H,k) local Ramsey number $r^{(H,k)}(G)$ is the smallest positive integer m such that each local (H,k)-coloring of K_m contains a monochromatic subgraph isomorphic to G.

Theorem 9. Let H be a graph with at least $k+1$ edges. The Ramsey number $r^{(H,k)}(G)$ is well defined for every graph G if and only if H contains a forest with $k+1$ edges. In such a case $r^{(H,k)}(G) \le (2+6k^3)(3k+1)^2 r_{loc}^k(G)$.

Let $E(K_n)$ be colored such that at least three edges have different colors. Then K_n contains a K_4 with three of its edges of different colors. Reason:



Thus it follows that each (K_{2k},k)-coloring of K_n is a k-coloring. Hence for $k \geq 2$ and for each graph G
$r^{(K_{2k},k)}(G) = r^k(G)$.

Theorem 10. Let $k \geq 1$ and $m = \lceil 3k/2 \rceil + 1$. Then for each connected graph G
$r^{(K_m,k)}(G) = r^k(G)$.

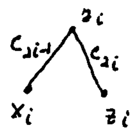
Proof. We need show $r^{(K_m,k)}(G) \leq r^k(G)$. Suppose this not case. Set

$n = r^k(G)$ and let ϕ be a (K_m,k)-coloring of K_n with no mono. G. If c_1 and c_2 are two colors of ϕ such that no pair of edges with these two colors are adjacent, then recolor K_n changing each edge colored c_2 to color c_1. Clearly the recolored K_n is an (K_m,k)-coloring with no mono. G (since G is connected)..

Repeat this recoloring procedure until an (K_m,k)-coloring ψ of K_n is obtained in which each pair of colors appear at least once as adjacent edges.

By assumption ψ uses at least

$k+1$ colors. If $k+1$ is even select any $k+1$ colors $c_1, c_2, \cdots, c_{k+1}$ used by ψ, and if $k+1$ is odd select three colors a, b, c appearing on edges of some K_4 and select the remaining $k-2$ colors $c_1, c_2, \cdots, c_{k-2}$ arbitrarily. Next choose vertices x_i, y_i, z_i such that



for $1 \leq i \leq \frac{k+1}{2}$ ($k+1$ even)
$1 \leq i \leq \frac{k-2}{2}$ ($k+1$ odd)

Then the set of chosen vertices X is such that $|X| \leq \lceil \frac{3k}{2} \rceil + 1 = m$. Hence ψ colors K_X with at least $k+1$ colors, a contradiction.

Question: What is the smallest value of $m \geq k+2$ such that for connected G, $r^{(K_m,k)}(G) = r^k(G)$? More generally what are the minimal graphs H containing a forest on $k+1$ edges such that for every graph G $r^{(H,k)}(G) = r^k(G)$?

Theorem 11. Let F be a forest with $k+1$ edges. For every graph G there exists a graph H such that the maximum clique size of H and G are the same and every (F,k)-coloring of H contains an induced monochromatic subgraph isomorphic to G.

Special (H,k) - Colorings

Let H be a graph containing at least $k+1$ edges. We are interested in those (H,k)-colorings of K_n $(n \geq n_0)$ such that each (H,k)-coloring is a k-coloring. It was observed earlier that (K_{2k},k)-colorings were such colorings. Whenever each (H,k)-coloring of K_n is a k-coloring we will call H a <u>k-good</u> graph.

<u>Problem</u>. Find necessary and sufficient conditions for a graph H to be k-good.

<u>Theorem 12.</u> If H is a k-good graph, then H contains each k edge graph as a subgraph.

<u>Proof</u>. Suppose not and assume H fails to contain some k edge graph L as a subgraph. Consider a fixed copy of L contained in K_n. Color each edge of this copy of L with a different color (colors $1, 2, \ldots, k$). Color all remaining edges of K_n with a $(k+1)$-st color. Clearly each copy of H in K_n is colored with at most k different colors (it must fail to contain some edge of L). Thus K_n has been given an (H,k)-coloring with $k+1$ colors, a contradiction.

<u>Conjecture</u>. The graph H containing at least $k+1$ edges is k-good if and only if H contains each k edge graph as a subgraph.

<u>Theorem 13</u>. The conjecture holds when

(i) $k = 2, 3,$ and 4,

(ii) H is the vertex disjoint union of all connected graphs on k edges $(k \geq 3)$.

(iii) $H =$  , i.e. H is obtained from K_k by attaching a pendant edge to each of its k vertices.

<u>Theorem 14</u>. The only edge minimal

(i) 2-good graphs are $P_4 =$ and $P_3 \cup P_2 =$

(ii) The only edge minimal 3-good graphs are

<u>Problem</u>. Which k edge graphs must H contain in order that under all (H,k)-colorings of K_n at most a bounded number of edges of K_n are colored differently?

We show $H \supseteq K_{1,k}, kK_2$

Reason:

(1) Color all edges of a fixed $K_{1,n-1}$ in K_n differently and all the remaining edges in K_n with a fixed color. Let H = union (vertex disjoint) of all connected k edge graphs (except for the star $K_{1,k}$). Clearly this an (H,k)-coloring of K_n which uses $n-1$ colors

(2) Next color $\lfloor \frac{n}{2} \rfloor$ independent edges of K_n differently and the other edges of K_n with a single color. In this case let $H = K_{2k-1}$. Again this is an (H,k)-colouring of K_n and contains $\lfloor \frac{n}{2} \rfloor$ colors. $\therefore H \supseteq K_{1,k}, kK_2$

Theorem 15. Let H be a graph with at least $k+1$ edges such that $H \supseteq K_{1,k}, kK_2$ (as subgraphs). If ϕ is an (H,k)-coloring of K_n, then K_n is colored by at most $3k^2$ colors.

Note k^2 is the correct order of magnitude. This seen by coloring each edge of a fixed copy of K_k in K_n differently and the remaining edges of K_n with a single color.

Let $H = K_{1,k} \cup kK_2$ and observe that this is an (H,k)-colouring of K_n with $\binom{k}{2}+1$ colors.

Questions

Local to Global Colorings

1) If H contains all k-edge graphs as subgraphs, then is H k-good?

2) Can one prove question raised above (in (1)) for special families of graphs H which contain all k-edge graphs?

3) What happens to the bound ck^2 of Theorem 14 when we assume H contains both $K_{1,k}$ and kK_2 and some of the other k edge graphs as subgraphs? How many of the k edge graphs must H contain for the bound to be linear in k?

Ramsey Questions

1) For which graphs H does $r^{(H,k)}(G) = r^k(G)$ for all graphs G?

2) If $r^{(K_m,k)}(G) = r^k(G)$, G connected, but $r^{(K_{m-1},k)}(G) > r^k(G)$, how large is this difference?

3) The size Ramsey number is defined as

$\hat{r}(G) = \min \{|E(H)| : H \to (G,G) \text{ minimally}\}$

Investigate the local size Ramsey number $\hat{r}^k_{loc}(G)$ and more generally $\hat{r}^{(H,k)}(G)$.

The Local Ramsey Number and Local Colorings

by

R. H. Schelp

REFERENCES

1) A. Gyárfás, J. Lehel, Zs. Tuza, R.H. Schelp, Ramsey Numbers for Local Colorings, Graphs and Comb. 3 (1987) 267-277.

2) A. Gyárfás, J. Lehel, J. Nešetřil, V. Rödl, R.H. Schelp, Zs. Tuza, Local k-Colorings of Graphs and Hypergraphs, JCTB 43(2) (1987) 127-139.

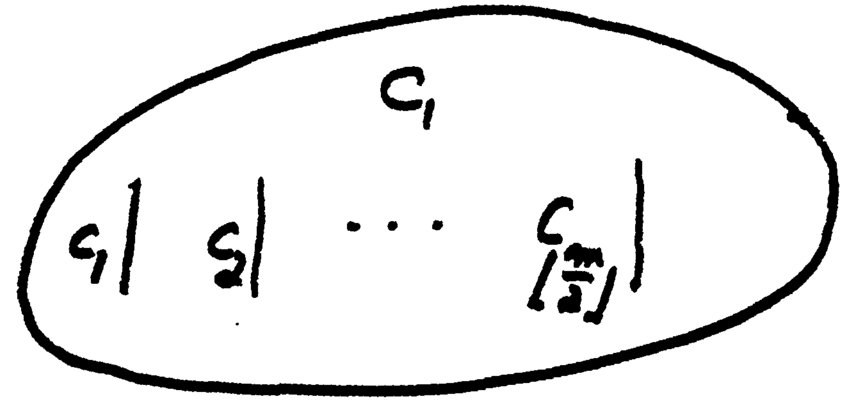
3) M. Truszczyński, Generalized Local Colorings of Graphs, to appear in JCTB.

4) R.A. Clapsaddle and R.H. Schelp, Local Edge Colorings that are Global, in preparation.

Def. A *local k-coloring* of a graph H is a coloring of its edges in such a way that the edges incident to each vertex of H are colored by at most k different colors. The *local Ramsey number* $r_{loc}^{k}(G)$ of a graph G is the smallest positive integer n such that K_n contains a monochromatic copy of G under each local k-coloring.

Observations:

1) For k fixed and m large a local k-coloring of K_m can use a large number of colors independent of k.

$K_m =$ 

Here all edges of K_m are colored with color c_1 except for the shown $\lfloor \frac{m}{2} \rfloor$ matching edges.

2) If $r^k(G)$ denotes the usual Ramsey number, then $r^k(G) \leq r^k_{loc}(G)$ since each coloring of a K_m by at most k colors is a local k-coloring.

Theorem 1 (Existence) For each $n \geq 3$, $k \geq 2$ $\Omega^k_{loc}(G) \leq \left\lceil k^{k(n-2)+1} / (k-1) \right\rceil$

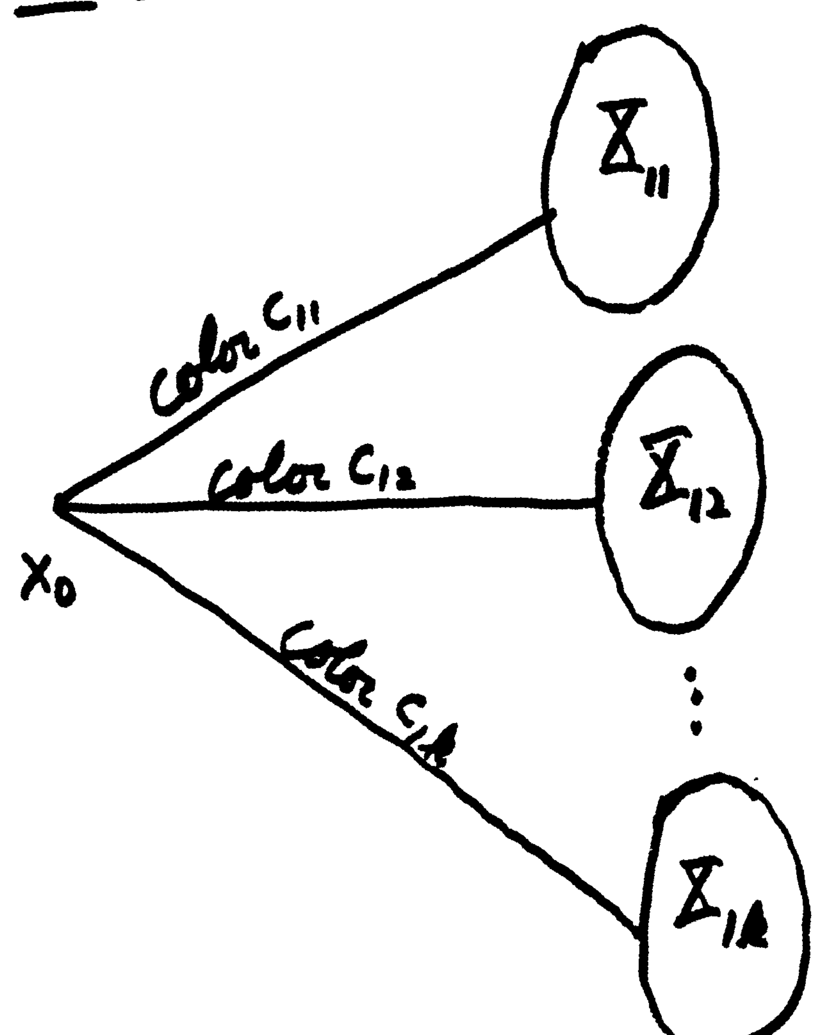
Proof. Let G be a local k-colored complete graph with $\geq \left\lceil k^{k(n-2)+1} / (k-1) \right\rceil$ vertices. We

show G contains a spanning tree T, rooted at a fixed vertex x_0, such that the following holds.

(i) Each $x \in V(T)$ has at most k successors $x_1, x_2, \cdots, x_s$ $(s \leq k)$ with different colors assigned to each edge xx_i, $1 \leq i \leq s$.

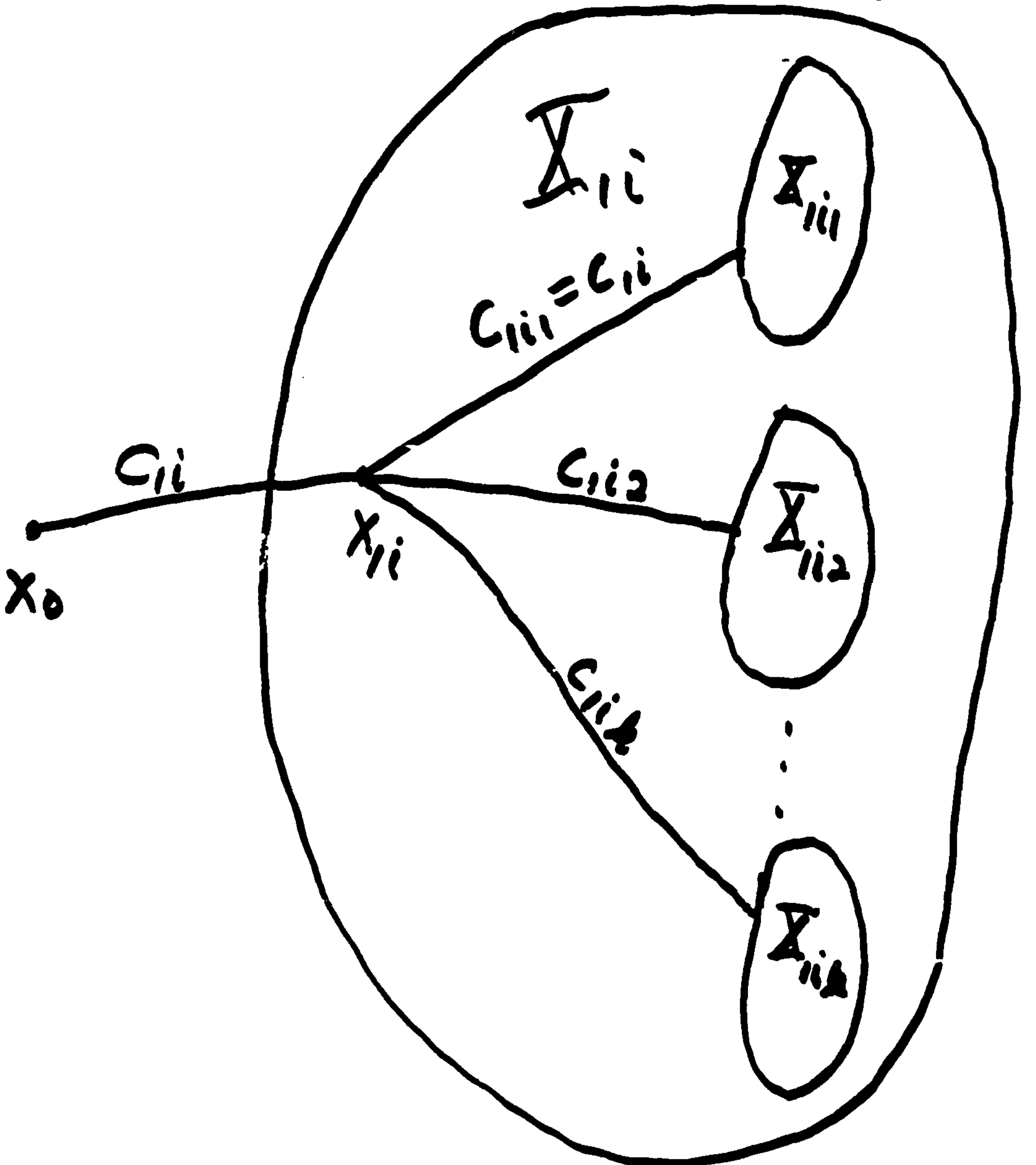
(ii) Edges xy and xz have the same color for $x, y, z \in V(T)$ when $x < y < z$, where the ordering corresponds to a partial order with x_0 as minimum element.

<u>How T is found</u>



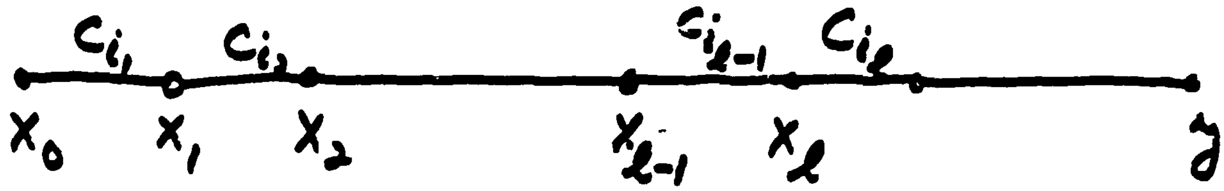
a partition of $V(G)$.

Select a fixed vertex x_{1i} in each X_{1i} and partition X_{1i} as was just done for $V(G)$ replacing x_0 by x_{1i}.

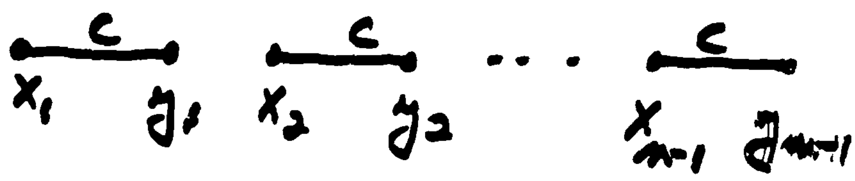


Continue this process until a spanning tree is obtained.

By condition (i) there exists a path P from the root x_0 to a vertex y with at least $k(n-2)+1$ edges.

P 

By condition (ii) x_ℓ is incident to edges colored $c_{i_1}, c_{i_2}, \ldots, c_{i_\ell}$. Therefore the edges of P get at most k colors. Hence there exist $n-1$ edges, say $x_1 y_1, x_2 y_2, \ldots, x_{n-1} y_{n-1}$ with the same color.



Then by (ii) $\{x_1, x_2, \ldots, x_{n-1}, y_{n-1}\}$ spans a K_n in color c.

Theorem 2

(i) $r^2_{loc}(K_n) = r^2(K_n)$ for all n.

(ii) $r^2_{loc}(K_{n-m} + \overline{K}_m) = r^2(K_{n-m} + \overline{K}_m)$,
$(n,m) \neq (3,2)$, $n \geq 2m-1$.

(iii) $r^2_{loc}(C_n) = r^2(C_n)$ for all $n \geq 3$.

(iv) $r^2_{loc}(P_{2m}) = r^2(P_{2m}) = 3m-1$, $m \geq 1$

(v) $r^2_{loc}(P_{2m+1}) = r^2(P_{2m+1}) + 1 = 3m+1$,
$m \geq 1$

(vi) $r^2_{loc}(nK_3) = 7n-2$, $n \geq 2$,
$r^2(nK_3) = 5n$, $n \geq 2$.

(vii) For a connected graph G
$r^2_{loc}(G) \geq 3|V(G)|/2$ and there are trees T such that $r^2(T) \leq \lfloor 4/3 \, |V(T)| - 1 \rfloor$

Theorem 3.

(i) $r^k_{loc}(S_n) = k(n-1)+2$, $k, n \geq 1$,
where S_n is star on n edges.

(ii) $r^k_{loc}(P_4) = \begin{cases} 2k+2 & \text{if } k \equiv 0, 1 \pmod 3 \\ 2k+1 & \text{if } k \equiv 2 \pmod 3 \end{cases}$

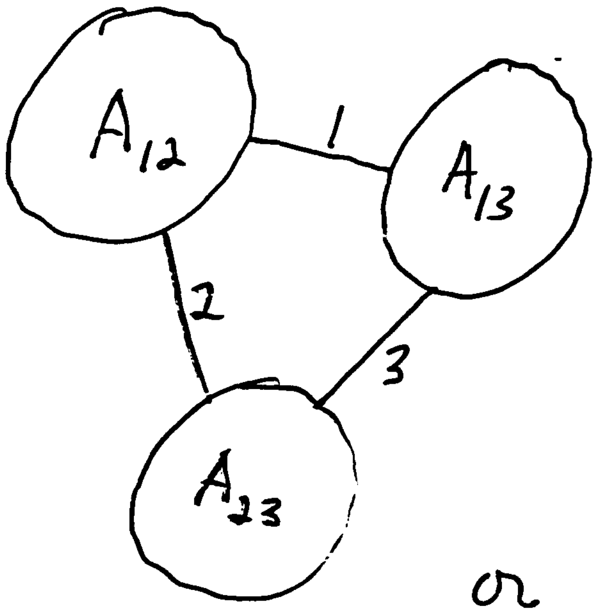
(iii) $r^3_{loc}(K_3) = 17$

(iv) (Truszczynski–Tuza)
For G a connected graph
$r^k_{loc}(G) / r^k(G) \leq c_k$ (c_k – constant depending on k)

(v) For $n \geq 1$, $t \geq 2$

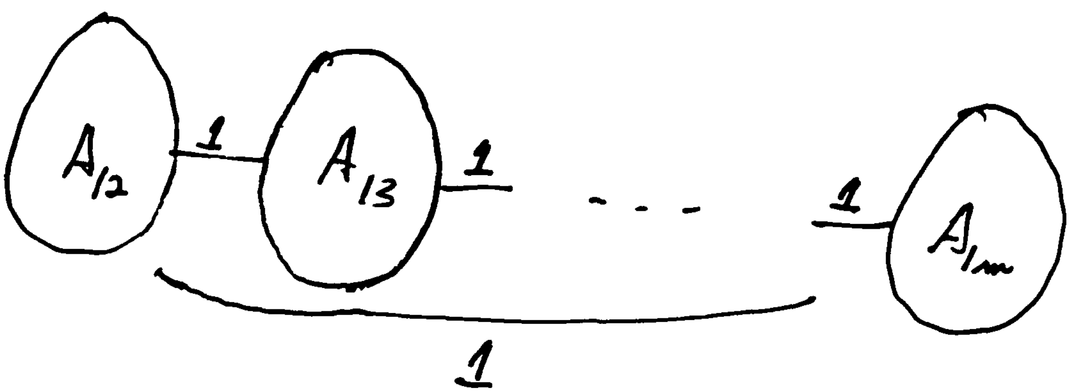
$r^2_{loc}(nK_t) \geq n(t^2 - t + 1) - t + 1$ and
for n large WRT t
$r^2(nK_t) \leq (2t-1)n + c_t$

Reason local 2-coloring problem is easier than for arbitrary k: Any local 2-coloring of K_n looks like



All edges in A_{ij} colored with either color i or color j.

or



Nature of local k-colorings

Theorem 4. If G is a locally k-colored graph, then for some monochromatic subgraph G_i, the average degree $d^*(G_i) \geq d^*(G)/k$.

Corollary. If G is locally k-colored, then it contains a monochromatic subgraph of minimum degree $\geq d^*(G)/2k$.

It is easy to see that if the edges of G are colored by at most k colors and $\chi(G) \geq m^k+1$, then G contains a monochromatic subgraph G' with $\chi(G') \geq m+1$.

This no longer holds for k-colorings.

Theorem 5. There exist graphs with arbitrary large chromatic number with local 2-colorings such that each monochromatic graph is bipartite.

Def. Let K_n^r denote the complete r-uniform hypergraph. A local k-coloring of K_n^r is a coloring of its edges such that the set of edges containing any $(r-1)$-element subset of vertices are colored by at most k different colors.

Theorem 6. (Existence - Ramsey Number for Hypergraphs)

Let k, r, and n be positive integers, $r \leq n$. Then there exists an $N = N(k, r, n)$ such that every local k-coloring of K_N^r contains a monochromatic K_n^r.

As an application of Theorem 6 can prove the following theorem.

Theorem 7. For all bipartite graphs B and for all k there exists a bipartite graph B' such that when B' is locally k-colored, then it contains a monochromatic copy of B as an induced subgraph of B'.

Theorem 8. Let G be a graph on n vertices with $\Delta(G) \leq d$. Then for each k there exists a function $c = c(k, d)$ such that $r_{loc}^k(G) \leq cn$.

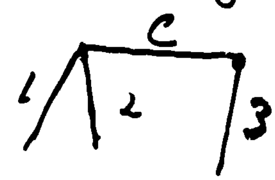
A Generalization by M. Truszczyński

<u>Def</u> Let k be a fixed positive integer and let H be a fixed graph with at least $k+1$ edges. We say a graph G has been given a <u>local</u> <u>(H, k)</u>-<u>Coloring</u> (or simply an (H,k)-coloring) if each subgraph of G isomorphic to H has its edges colored by at most k different colors.

Note that a local k-coloring of K_n is a $(K_{1,n-1}, k)$-coloring.

The <u>(H,k) local Ramsey number</u> $r^{(H,k)}(G)$ is the smallest positive integer m such that each local (H,k)-coloring of K_m contains a monochromatic subgraph isomorphic to G.

<u>Theorem 9.</u> Let H be a graph with at least $k+1$ edges. The Ramsey number $r^{(H,k)}(G)$ is well defined for every graph G if and only if H contains a forest with $k+1$ edges. In such a case $r^{(H,k)}(G) \leq (2+6k^2)(2k+1)^2 r^k_{loc}(G)$.

Let $E(K_n)$ be colored such that at least three edges have different colors. Then K_n contains a K_4 with three of its edges of different colors. Reason: 

Thus it follows that each (K_{2k}, k)-coloring of K_n is a k-coloring. Hence for $k \geq 2$ and for each graph G
$$r^{(K_{2k},k)}(G) = r^k(G).$$

<u>Theorem 10</u>. Let $k \geq 1$ and $m = \lceil 3k/2 \rceil + 1$. Then for each connected graph G
$$r^{(K_m,k)}(G) = r^k(G).$$

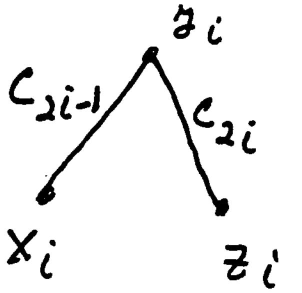
<u>Proof</u>. We need show $r^{(K_m,k)}(G) \leq r^k(G)$. Suppose this not case. Set

$n = r^k(G)$ and let ϕ be a (K_m, k)-coloring of K_n with no mono. G. If c_1 and c_2 are two colors of ϕ such that no pair of edges with these two colors are adjacent, then recolor K_n changing each edge colored c_2 to color c_1. Clearly the recolored K_n is an (K_m, k)-coloring with no mono. G (since G is connected)..

Repeat this recoloring procedure until an (K_m, k)-coloring ψ of K_n is obtained in which each pair of colors appear at least once as adjacent edges.

By assumption ψ uses at least

$k+1$ colors. If $k+1$ is even select any $k+1$ colors $c_1, c_2, \cdots, c_{k+1}$ used by φ, and if $k+1$ is odd select three colors a, b, c appearing on edges of some K_4 and select the remaining $k-2$ colors $c_1, c_2, \cdots, c_{k-2}$ arbitrarily. Next choose vertices x_i, y_i, z_i such that



for $1 \le i \le \frac{k+1}{2}$ ($k+1$ even)

$1 \le i \le \frac{k-2}{2}$ ($k+1$ odd)

Then the set of chosen vertices X is such that $|X| \le \left\lceil \frac{3k}{2} \right\rceil + 1 = m$.

Hence φ colors K_X with at least $k+1$ colors, a contradiction.

<u>Question</u>: What is the smallest value of $m \geq k+2$ such that for connected G, $r^{(K_m, k)}(G) = r^k(G)$? More generally what are the minimal graphs H containing a forest on $k+1$ edges such that for every graph G $r^{(H,k)}(G) = r^k(G)$?

<u>Theorem 11.</u> Let F be a forest with $k+1$ edges. For every graph G there exists a graph H such that the maximum clique size of H and G are the same and every (F,k)-coloring of H contains an induced monochromatic subgraph isomorphic to G.

Special (H,k) - Colorings

Let H be a graph containing at least $k+1$ edges. We are interested in those (H,k)-colorings of K_n $(n \geq n_0)$ such that each (H,k)-coloring is a k-coloring. It was observed earlier that (K_{2k},k)-colorings were such colorings. Whenever each (H,k)-coloring of K_n is a k-coloring we will call H a <u>k-good</u> graph.

<u>Problem</u>. Find necessary and sufficient conditions for a graph H to be k-good.

Theorem 12. If H is a k-good graph, then H contains each k edge graph as a subgraph.

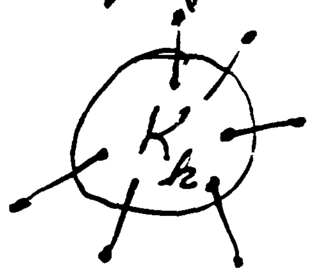
Proof. Suppose not and assume H fails to contain some k edge graph L as a subgraph. Consider a fixed copy of L contained in K_n. Color each edge of this copy of L with a different color (colors $1, 2, \ldots, k$). Color all remaining edges of K_n with a $(k+1)$-st color. Clearly each copy of H in K_n is colored with at most k different colors (it must fail to contain some edge of L). Thus K_n has been given an (H,k)-coloring with $k+1$ colors, a contradiction.

<u>Conjecture</u>. The graph H containing at least $k+1$ edges is k-good if and only if H contains each k edge graph as a subgraph.

<u>Theorem 13</u>. The conjecture holds when

(i) $k = 2, 3,$ and 4,

(ii) H is the vertex disjoint union of all connected graphs on k edges ($k \geq 3$).

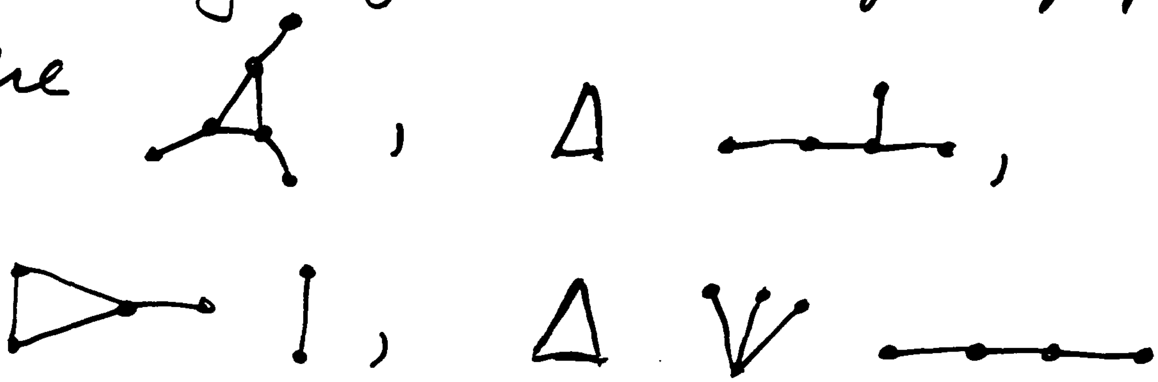
(iii) $H =$ , i.e. H is obtained from K_k by attaching a pendant edge to each of its k vertices.

Theorem 14. The only edge minimal

(i) 2-good graphs are $P_4 =$ 

and $P_3 \cup P_2 =$ 

(ii) The only edge minimal 3-good graphs are



Problem. Which k edge graphs must H contain in order that under all (H, k)-colourings of K_n at most a bounded number of edges of K_n are colored differently?

We show $H \supseteq K_{1,k}, kK_2$

Reason:

(1) Color all edges of a fixed $K_{1,n-1}$ in K_n differently and all the remaining edges in K_n with a fixed color. Let H = union (vertex disjoint) of all connected k edge graphs (except for the star $K_{1,k}$). Clearly this an (H,k)-coloring of K_n which uses $n-1$ colors.

(2) Next color $\lfloor \frac{n}{2} \rfloor$ independent edges of K_n differently and the other edges of K_n with a single color. In this case let $H = K_{2k-1}$. Again this is an (H,k)-coloring of K_n and contains $\lfloor \frac{n}{2} \rfloor$ colors. $\therefore H \supseteq K_{1,k}, kK_2$

Theorem 15. Let H be a graph with at least $k+1$ edges such that $H \supseteq K_{1,k}$, kK_2 (as subgraphs). If ϕ is an (H,k)-coloring of K_n, then K_n is colored by at most $3k^2$ colors.

Note k^2 is the correct order of magnitude. This seen by coloring each edge of a fixed copy of K_k in K_n differently and the remaining edges of K_n with a single color.

Let $H = K_{1,k} \cup kK_2$ and observe that this is an (H,k)-colouring of K_n with $\binom{k}{2}+1$ colors.

Questions

Local to Global Colorings

1) If H contains all k-edge graphs as subgraphs, then is H k-good?

2) Can one prove question raised above (in (1)) for special families of graphs H which contain all k-edge graphs?

3) What happens to the bound ck^2 of Theorem 14 when we assume H contains both $K_{1,k}$ and kK_2 and some of the other k edge graphs as subgraphs? How many of the k edge graphs must H contain for the bound to be linear in k?

Ramsey Questions

1) For which graphs H does
$r^{(H,k)}(G) = r^{k}(G)$ for all graphs G?

2) If $r^{(K_m,k)}(G) = r^{k}(G)$, G connected, but $r^{(K_{m-1},k)}(G) > r^{k}(G)$, how large is this difference?

3) The size Ramsey number is defined as
$$\hat{r}(G) = \min\{|E(H)| : H \to (G,G) \text{ minimally}\}$$
Investigate the local size Ramsey number $\hat{r}^{k}_{loc}(G)$ and more generally $\hat{r}^{(H,k)}(G)$.

New algorithms for minimizing convex functions over convex sets

Prof. Pravin Vaidya
Department of Computer Science
University of Illinois

New algorithms for minimizing convex functions over convex sets*

Pravin M. Vaidya
Dept. of Computer Science,
Univ. of Illinois at Urbana-Champaign

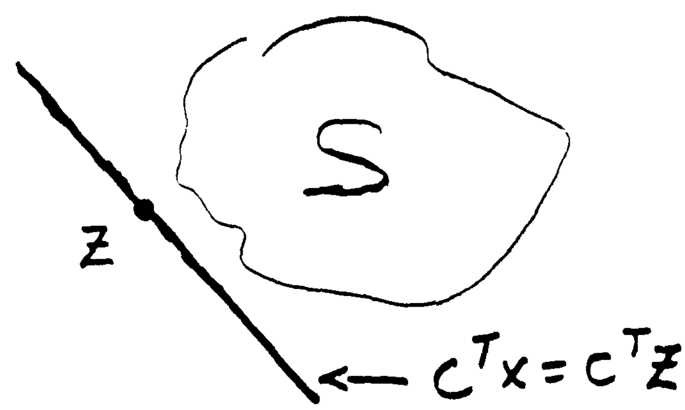
* Part of this work was done when the author was at AT&T Bell Laboratories, Murray Hill, NJ

Let $S \subseteq \mathbb{R}^n$ such that there is an oracle for S with the following property.

Let $z \in \mathbb{R}^n$ be a test point.

i) Oracle answers "Yes" if $z \in S$.

ii) Oracle returns a vector c such that $S \subseteq \{x : c^T x \geq c^T z\}$ if $z \notin S$.



<u>Feasibility Problem</u>

Find a point in S

<u>Optimization Problem</u>

Minimize a convex function over S.

Applications

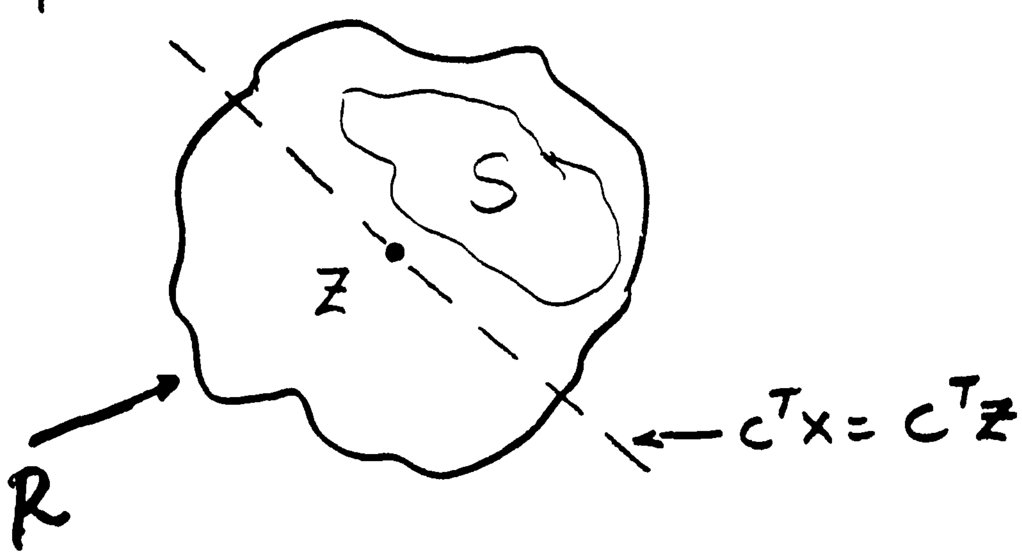
1. Econometric, statistical modelling
2. Structural Optimization
3. Relaxations of NP-hard problems
4. Certain non-linear PDE's
5. VLSI Design
6. Combinatorial Optimization

⋮

Feasibility Problem

i) Maintain a region R s.t. $S \subseteq R$.

ii) At each step select a test point $z \in R$ & call the oracle with z as input.

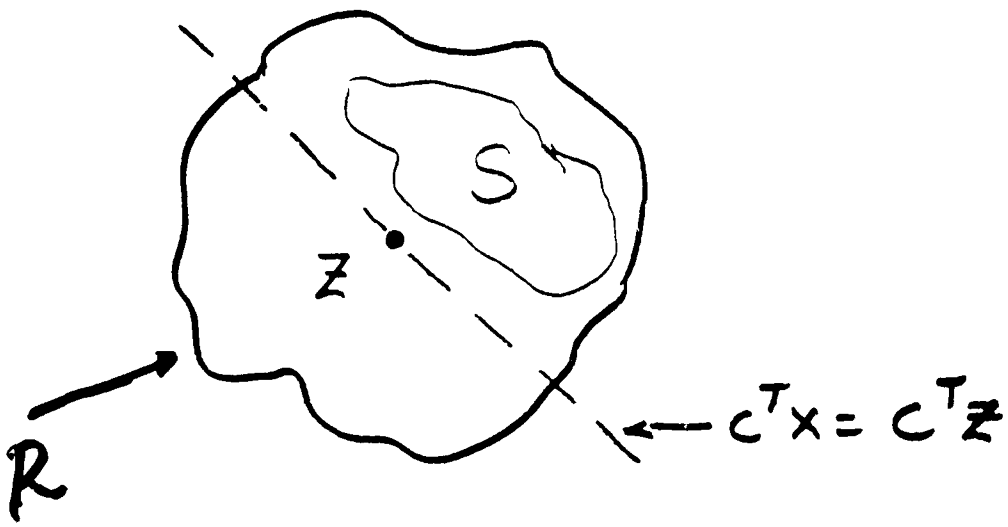


$$z \notin S \implies S \subseteq R \cap \{x : c^T x \geq c^T z\}$$

Feasibility Problem

i) Maintain a region R s.t. $S \subseteq R$

ii) At each step select a test point $z \in R$ & call the oracle with z as input.

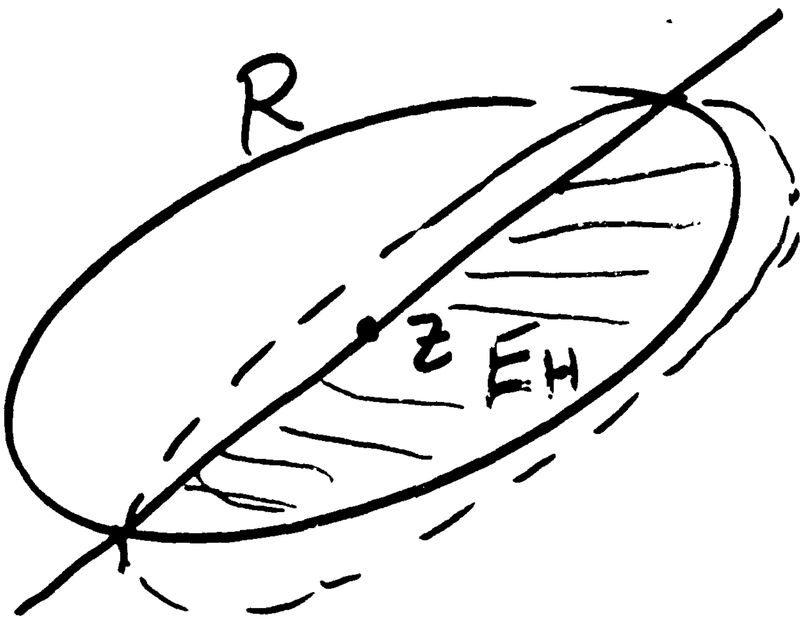


$$z \notin S \implies S \subseteq R \cap \{x : c^T x \geq c^T z\}$$

Ellipsoid algorithm

(i) Region R is an ellipsoid

(ii)



Z : center of ellipsoid R

Description of R is simplified at each step by redrawing an ellipsoid of smaller volume around half ellipsoid E_H

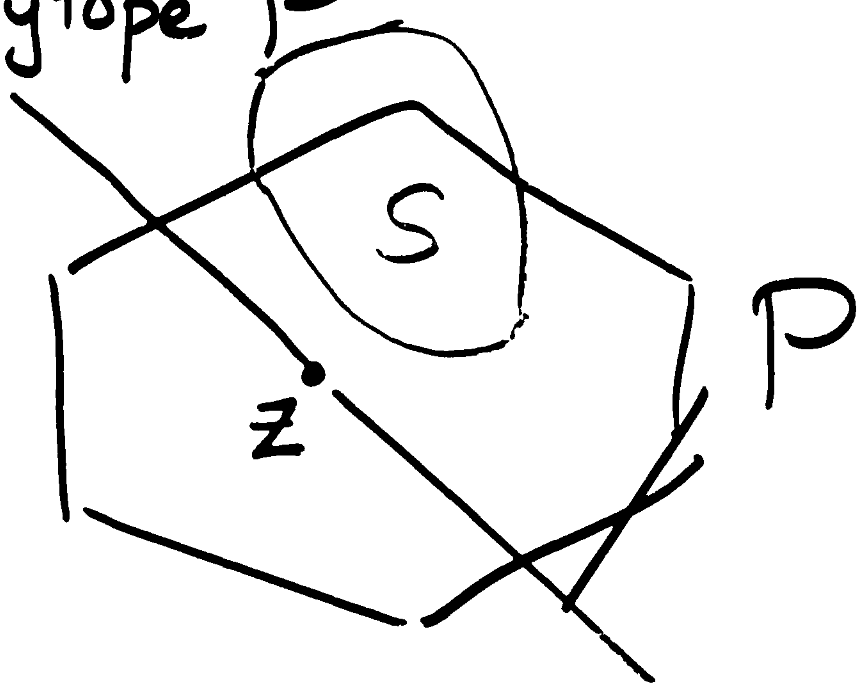
A class of algorithms based on polytopes

(i) Region R is a bounded full dimensional polytope

$$P = \{x : A x \geq b\}$$

$A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$

(ii) z is a suitable "center" (or a balanced point) in the polytope P



Possible choices for centers

(i) **Analytic center:**

Logarithmic barrier $\phi(x)$

$$\phi(x) = -\sum_{i=1}^{m} \ln(a_i^T x - b_i)$$

Analytic center is the minimizer of $\phi(x)$ over P.

(ii) **Volumetric center:**

Determinant Barrier $F(x)$

$$F(x) = \frac{1}{2}\ln\left(\det(\nabla^2\phi(x))\right)$$

(iii) Center that maximizes the volume of an ellipsoid inscribable in the polytope

(iv) Weighted analytic center

$$\text{logbar}(W,x) = -\sum_{i=1}^{m} w_i \ln(a_i^T x - b_i)$$

This center minimizes logbar(W, x)

(v) Center of gravity

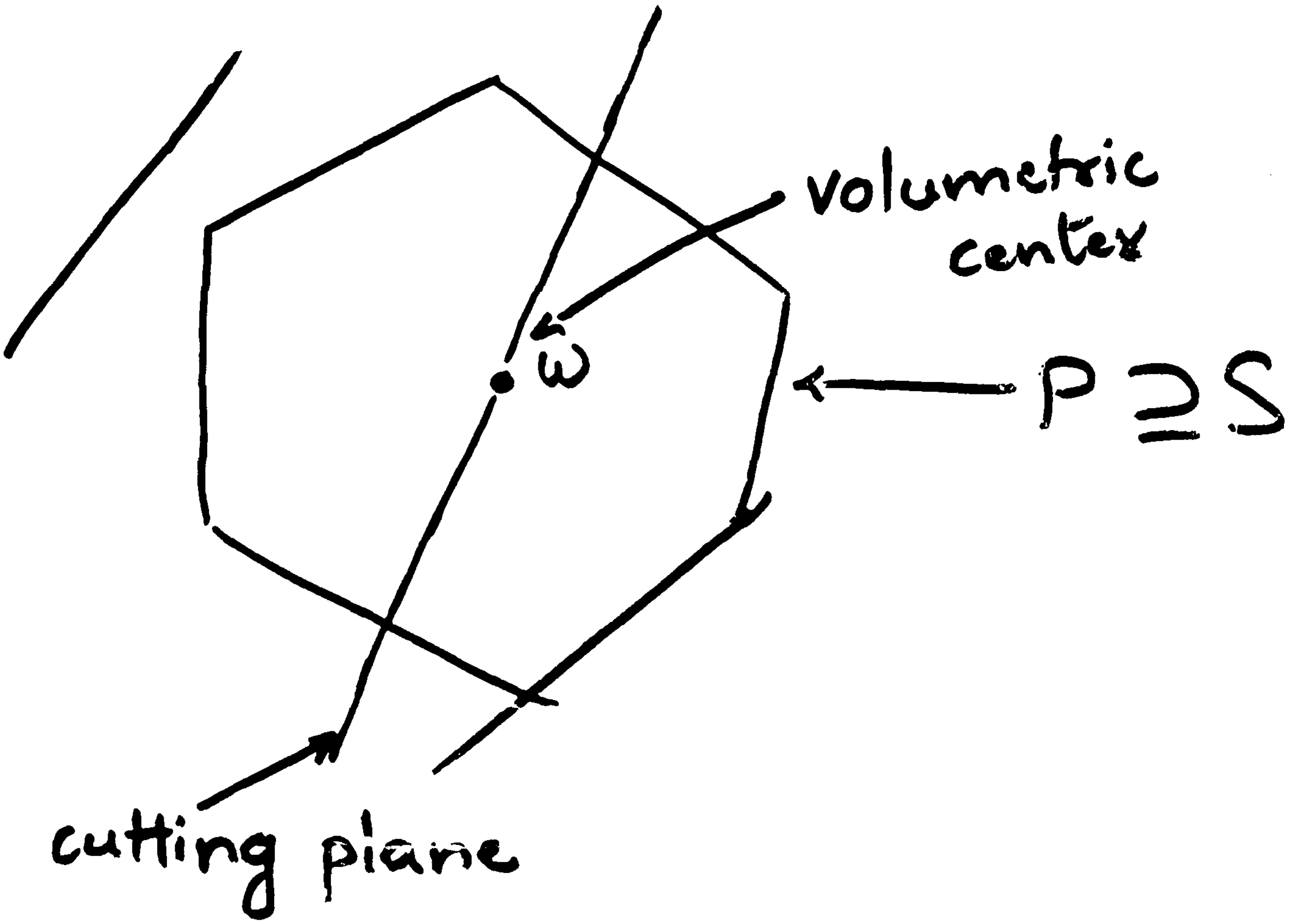
Algorithm for the feasibility problem

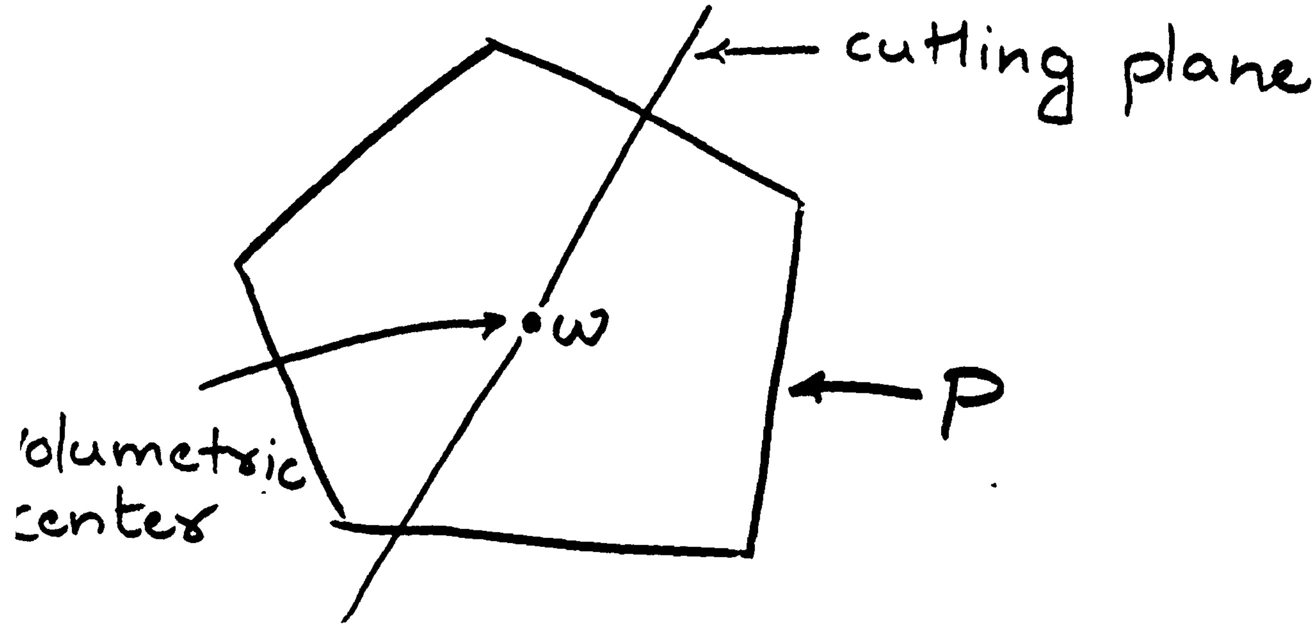
1) The region R is a polytope P,

$$P = \{x : Ax \geq b\}$$

P - full dimensional, bounded

2) Test point Z is the volumetric center of P





Volumetric center of P

$P = \{x : Ax \geq b\}, \quad A \in R^{m \times n}, \quad b \in R^m$

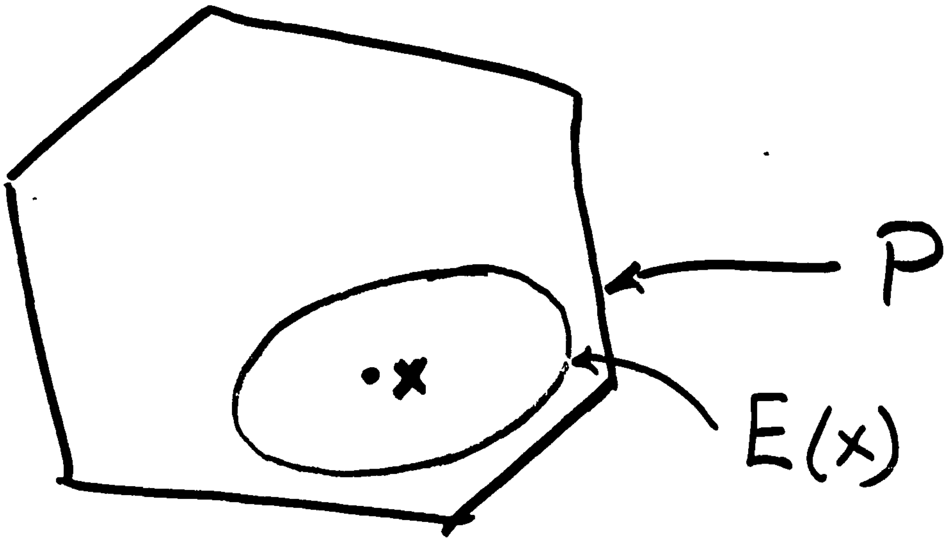
log barrier: $\phi(x) = -\sum_{i=1}^{m} \ln(a_i^T x - b_i)$

$$H(x) = \sum_{i=1}^{m} \frac{a_i a_i^T}{(a_i^T x - b_i)^2}$$

Volumetric center minimizes $\det(H(x))$ over P.

→ determinant of $H(x)$

Geometric Interpretation of the Volumetric center ω



$$E(x) = \{ y : (y-x)^T H(x)(y-x) \leq 1 \}$$

(i) $E(x) \subseteq P$

(ii) $E(\omega)$ has the largest volume among all ellipsoids $E(x)$

$E(\omega)$ is a maximum volume quadratic approximation to polytope P.

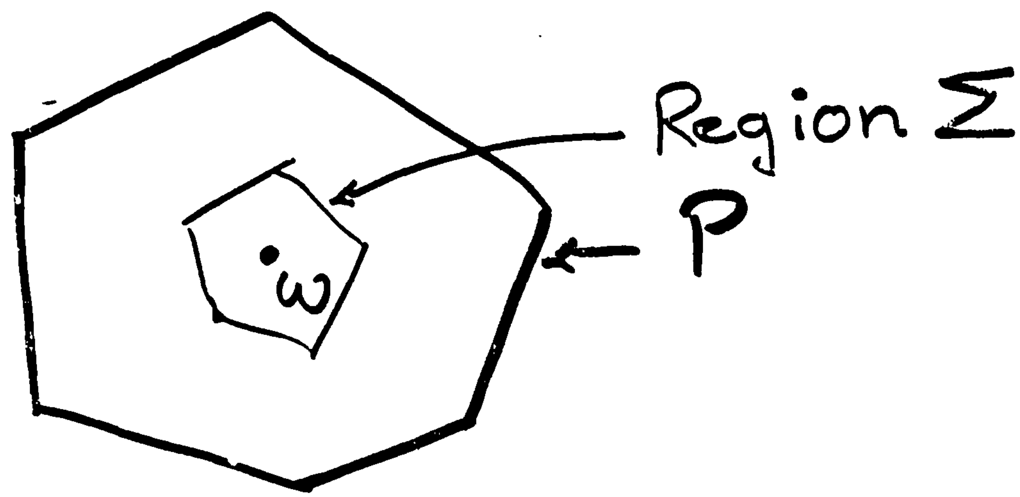
Computing the volumetric center

A step (Newton's Method)

Current point z

$$z \leftarrow z - t\, Q(z)^{-1} \nabla F(z)$$

t a suitable scalar



1) $z \in \Sigma$: $F(z) - F(\omega)$ decreases by a constant factor

2) $z \notin \Sigma$: $F(z)$ decreases by about $\frac{1}{\sqrt{m}}$

Finding the volumetric center ω

$$H(x) = \sum_{i=1}^{m} \frac{a_i a_i^T}{(a_i^T x - b_i)^2}$$

$$F(x) = \frac{1}{2} \ln(\det(H(x)))$$

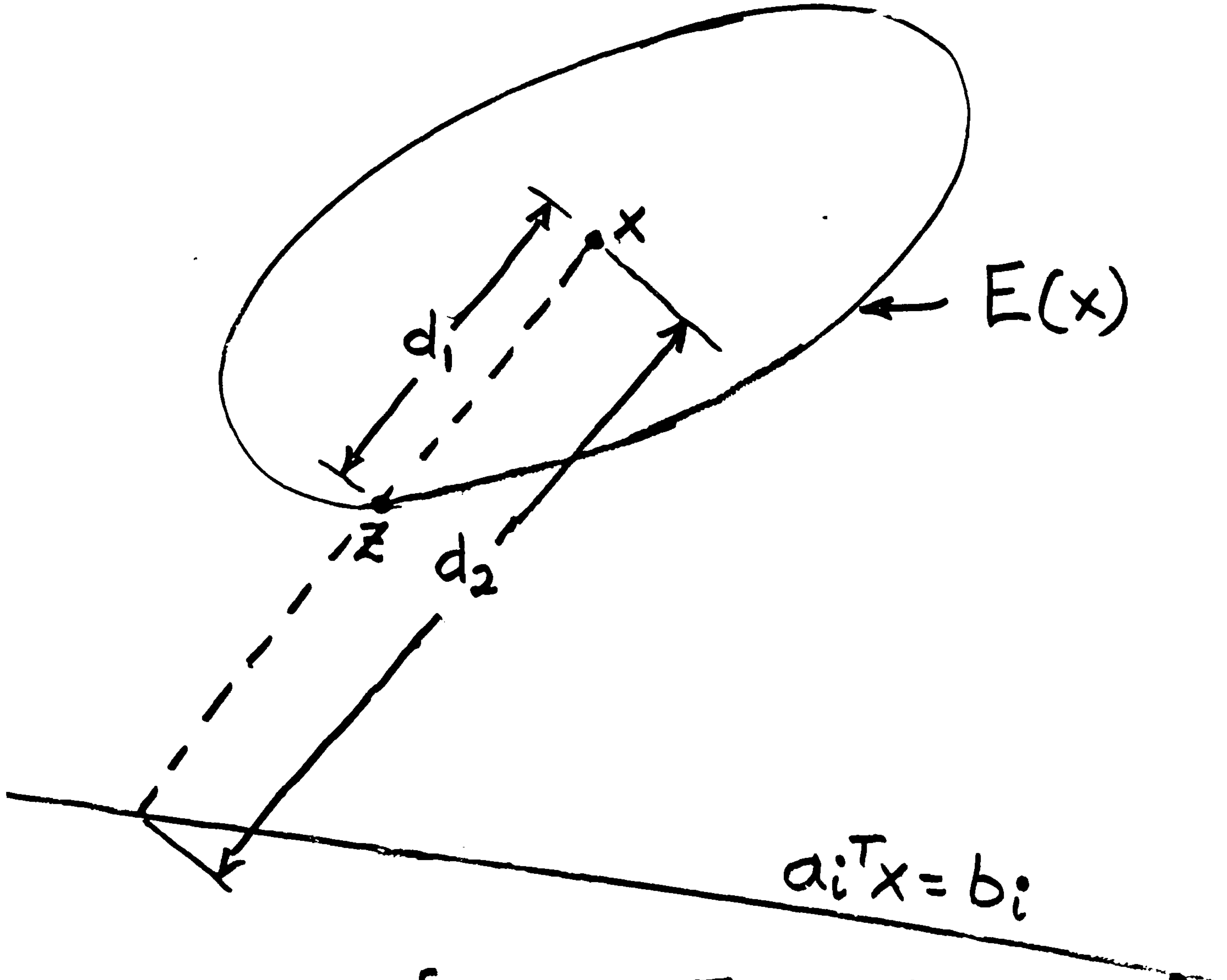
$$\sigma_i(x) = \frac{a_i^T H(x)^{-1} a_i}{(a_i^T x - b_i)^2}, \quad 1 \le i \le m$$

$$\nabla F(x) = -\sum_{i=1}^{m} \sigma_i(x) \frac{a_i}{a_i^T x - b_i}$$

$$Q(x) = \sum_{i=1}^{m} \sigma_i(x) \frac{a_i a_i^T}{(a_i^T x - b_i)^2}$$

$Q(x)$ approximates the Hessian $\nabla^2 F(x)$

Interpretation of weights $\sigma_i(x)$



$$E(x) = \{ y : (y-x)^T H(x) (y-x) \le 1 \}$$

z minimizes $a_i^T x$ over $E(x)$

$$\sigma_i(x) = \left(\frac{d_1}{d_2}\right)^2$$

Pruning the Polytope P

$P = \{x : Ax \geq b\}$

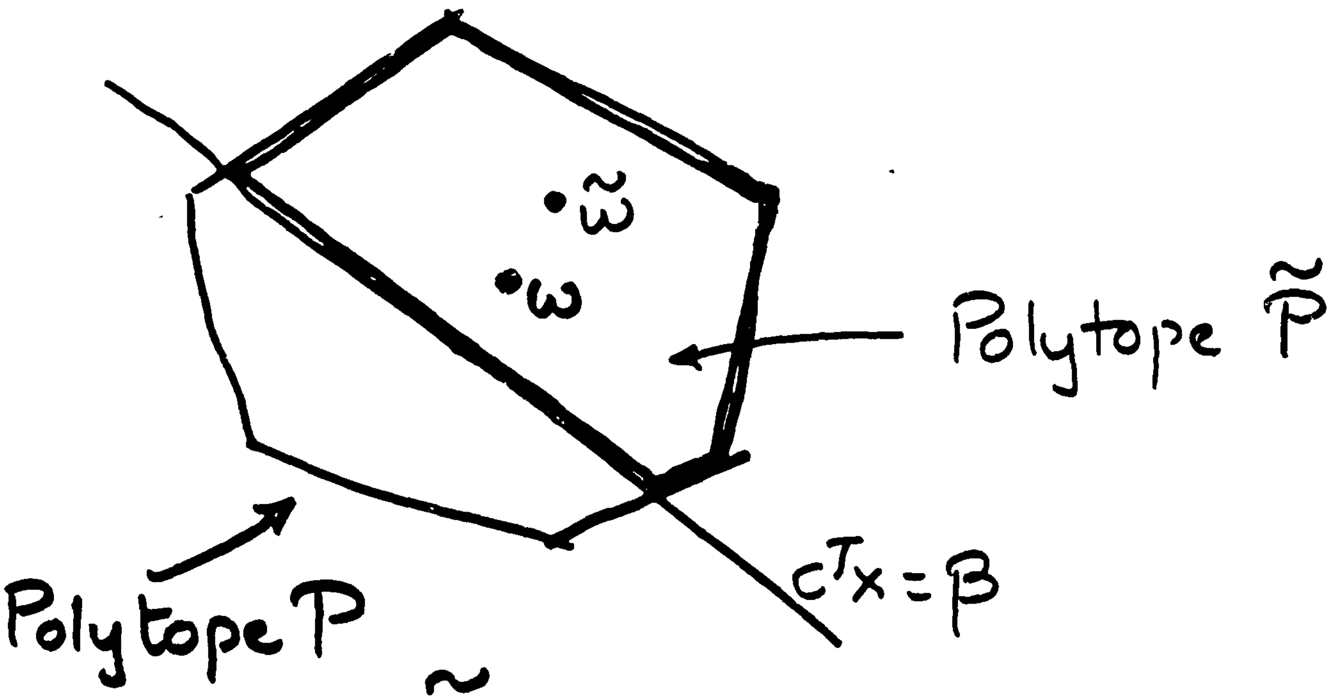
$A \in \mathbb{R}^{m \times n}$; m constraining planes

As m increases, the "centers" get unbalanced, convergence can slow down & computational work/step increases.

Polytope P may be pruned i.e. some of the planes defining P are dropped

$\sigma_i(x)$ small $\Longrightarrow$ $i^{\underline{th}}$ constraint $a_i^T x = b_i$ may be dropped

Cutting the polytope near the volumetric center



ⓐ $\tilde{P} = P \bigcap \{x : c^T x \geq \beta\}$

ⓑ $$\frac{c^T H(\omega)^{-1} c}{(c^T \omega - \beta)^2} = \frac{\alpha}{\sqrt{m}}$$

$$\tilde{F}(\tilde{\omega}) - F(\omega) \sim \frac{\alpha}{2\sqrt{m}}$$

Algorithm with best complexity

1) Maintain a polytope P such that $S \subseteq P$.
2) Use a good approximation to volumetric center as the test point
3) Also prune the polytope P i.e. drop some of the planes from time to time so $m = O(n)$

$F(w)$ increases by a fixed constant δ at each step & after K steps

$$\boxed{\text{volume}(P) \leq \left(\frac{n}{\delta}\right)^n e^{-k\delta}}$$

Variants of the algorithm

Desirable properties

(a) Computation at a step as simple as possible

Preferably a single linear system solve

(b) Exploit underlying structure of constraints defining S

eg. Constraints defining S may be explicitly given & each constraint depends only on a few variables.

(c) Polynomial convergence still maintained in the worst case

Possible directions for variants

1) Interpreting the volumetric center as a weighted analytic center and Dynamically weighting the planes
2) Combination of determinant barrier & logarithmic barrier
3) Combination of determinant barriers
4) Several mildly non-linear functions together with a few highly non-linear functions

Several mildly non-linear fns. together with a few highly non-linear ones

$$\max \; p^T x$$

$$\text{s.t. } g_i(x) \geq 0, \; 1 \leq i \leq m$$

Most of g_i's are only mildly non-linear, g_i's are concave.

$$\phi(\beta, x) = m \ln(p^T x - \beta) + \sum_{i=1}^{m} \ln(g_i(x))$$

Related centering problem

Compute maximizer of $\phi(\beta, x)$

"Lazy use of separating tangent planes"

Centering problem

maximize $\phi(\beta, x)$ where

$$\phi(\beta, x) = m \ln(p^T x - \beta) + \sum_{i=1}^{m} \ln(g_i(x))$$

Alternate between Newton's method & a method that is based on separating (tangent) planes; the subroutine based on separating planes is called only when Newton's method fails to make progress in a consecutive number of steps.

Applications to linear programming

1) The basic algorithm or any suitable variant can solve a linear program with exponentially many constraints as long as there is a good subroutine to generate violated constraints.

Examples — LP relaxations of TSP & maximum independent set.

Weighted matching

2) Possible dynamic weighting of planes in ordinary linear programming.

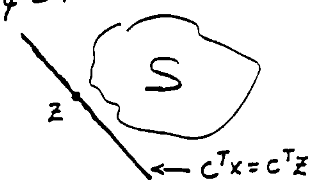
New algorithms for minimizing convex functions over convex sets*

Pravin M. Vaidya
Dept. of Computer Science,
Univ. of Illinois at Urbana-Champaign

* Part of this work was done when the author was at AT&T Bell Laboratories, Murray Hill, NJ

Let $S \subseteq \mathbb{R}^n$ such that there is an oracle for S with the following property.
Let $z \in \mathbb{R}^n$ be a test point.

i) Oracle answers "Yes" if $z \in S$.
ii) Oracle returns a vector c such that $S \subseteq \{x : c^T x \geq c^T z\}$ if $z \notin S$.



Feasibility Problem
Find a point in S

Optimization Problem
Minimize a convex function over S.

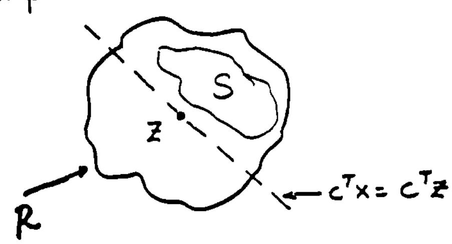
Applications

1. Econometric, statistical modelling
2. Structural Optimization
3. Relaxations of NP-hard problems
4. Certain non-linear PDE's
5. VLSI Design
6. Combinatorial Optimization

⋮

Feasibility Problem

i) Maintain a region R s.t. $S \subseteq R$.
ii) At each step select a test point $z \in R$ & call the oracle with z as input.

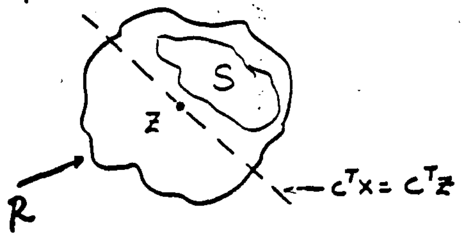


$$z \notin S \implies S \subseteq R \cap \{x : c^T x \geq c^T z\}$$

Feasibility Problem

i) Maintain a region R s.t. $S \subseteq R$

ii) At each step select a test point $z \in R$ & call the oracle with z as input.

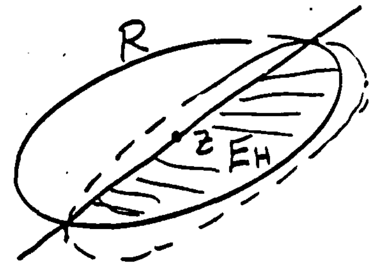


$$z \notin S \implies S \subseteq R \cap \{x : c^Tx \geq c^Tz$$

Ellipsoid algorithm

(i) Region R is an ellipsoid

(ii)



z: center of ellipsoid R

Description of R is simplified at each step by redrawing an ellipsoid of smaller volume around half ellipsoid E_H

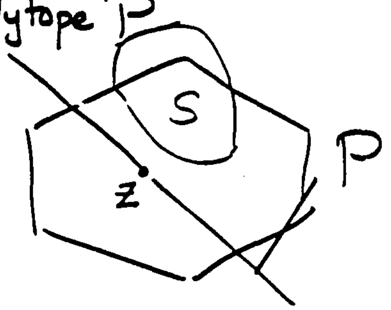
A class of algorithms based on polytopes

(i) Region R is a bounded full dimensional polytope

$$P = \{x : Ax \geq b\}$$

$$A \in \mathbb{R}^{m \times n}; \; b \in \mathbb{R}^m$$

(ii) z is a suitable "center" (or a balanced point) in the polytope P



Possible choices for centers

(i) Analytic center:

Logarithmic barrier $\phi(x)$

$$\phi(x) = -\sum_{i=1}^{m} \ln(a_i^Tx - b_i)$$

Analytic center is the minimizer of $\phi(x)$ over P.

(ii) Volumetric center:

Determinant Barrier $F(x)$

$$F(x) = \frac{1}{2} \ln\left(\det(\nabla^2 \phi(x))\right)$$

(iii) Center that maximizes the volume of an ellipsoid inscribable in the polytope

(iv) Weighted analytic center

$$\operatorname{logbar}(W,x) = -\sum_{i=1}^{m} w_i \ln(a_i^T x - b_i)$$

This center minimizes $\operatorname{logbar}(W,x)$

(v) Center of gravity

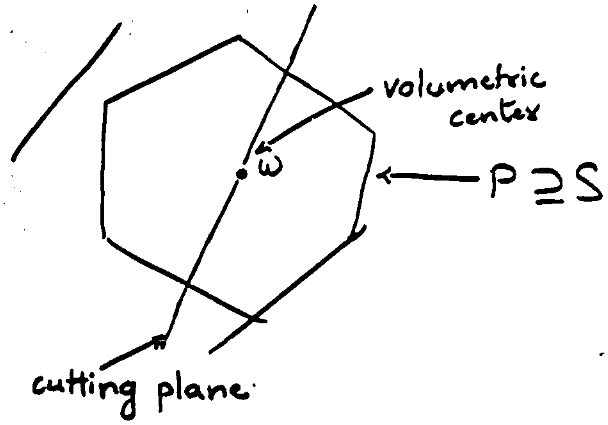
Algorithm for the feasibility problem

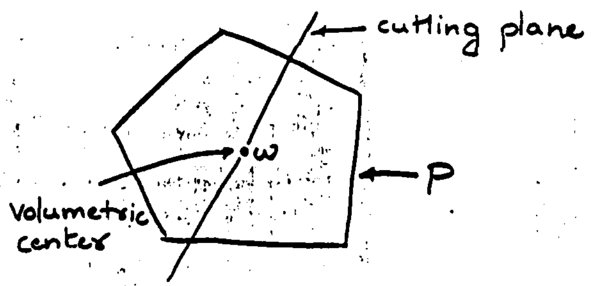
1) The region R is a polytope P,

$$P = \{x : Ax \geq b\}$$

P – full dimensional, bounded

2) Test point z is the volumetric center of P





Volumetric center of P

$P = \{x : Ax \geq b\}$, $A \in R^{m \times n}$, $b \in R^m$

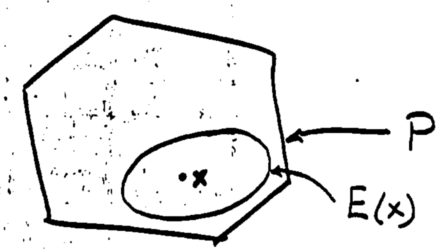
log barrier: $\phi(x) = -\sum_{i=1}^{m} \ln(a_i^T x - b_i)$

$$H(x) = \sum_{i=1}^{m} \frac{a_i a_i^T}{(a_i^T x - b_i)^2}$$

Volumetric center minimizes $\det(H(x))$ over P.

→ determinant of $H(x)$

Geometric Interpretation of the Volumetric center ω



$$E(x) = \{y : (y-x)^T H(x)(y-x) \leq 1\}$$

(i) $E(x) \subseteq P$

(ii) $E(\omega)$ has the largest volume among all ellipsoids $E(x)$

$E(\omega)$ is a maximum volume quadratic approximation to polytope P.

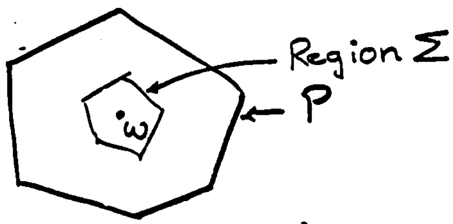
Computing the volumetric center

A step (Newton's Method)

Current point z

$$z \longleftarrow z - t\, Q(z)^{-1} \nabla F(z)$$

t a suitable scalar



1) $z \in \Sigma$: $F(z) - F(w)$ decreases by a constant factor

2) $z \notin \Sigma$: $F(z)$ decreases by about $\frac{1}{\sqrt{m}}$

Finding the volumetric center w

$$H(x) = \sum_{i=1}^{m} \frac{a_i a_i^T}{(a_i^T x - b_i)^2}$$

$$F(x) = \frac{1}{2} \ln\left(\det(H(x))\right)$$

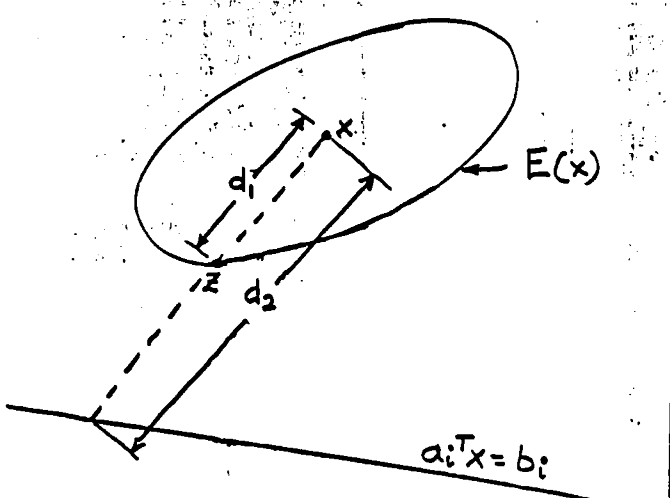
$$\sigma_i(x) = \frac{a_i^T H(x)^{-1} a_i}{(a_i^T x - b_i)^2}, \quad 1 \le i \le m$$

$$\nabla F(x) = -\sum_{i=1}^{m} \sigma_i(x) \frac{a_i}{a_i^T x - b_i}$$

$$Q(x) = \sum_{i=1}^{m} \sigma_i(x) \frac{a_i a_i^T}{(a_i^T x - b_i)^2}$$

$Q(x)$ approximates the Hessian $\nabla^2 F(x)$

Interpretation of weights $\sigma_i(x)$



$$E(x) = \{ y : (y-x)^T H(x) (y-x) \le 1 \}$$

z minimizes $a_i^T x$ over $E(x)$

$$\sigma_i(x) = \left(\frac{d_1}{d_2}\right)^2$$

Pruning the Polytope P

$P = \{x : Ax \ge b\}$

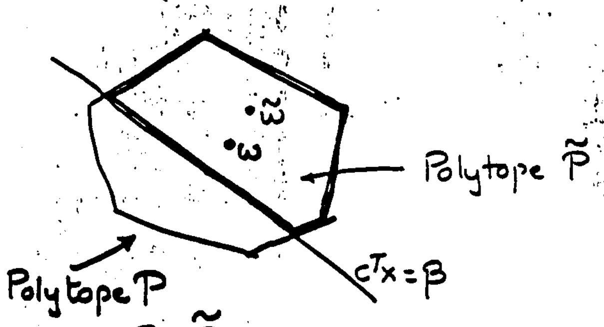
$A \in \mathbb{R}^{m \times n}$; m constraining planes

As m increases, the "centers" get unbalanced, convergence can slow down & computational work/step increases.

Polytope P may be pruned i.e. some of the planes defining P are dropped

$\sigma_i(x)$ small $\Rightarrow$ i^{th} constraint $a_i^T x = b_i$ may be dropped

Cutting the polytope near the volumetric center



(a) $\tilde{P} = P \cap \{x : c^T x \geq \beta\}$

(b) $$\frac{c^T H(\omega)^{-1} c}{(c^T\omega - \beta)^2} = \frac{\alpha}{\sqrt{m}}$$

$$\tilde{F}(\tilde{\omega}) - F(\omega) \sim \frac{\alpha}{2\sqrt{m}}$$

Algorithm with best complexity

1) Maintain a polytope P such that $S \subseteq P$.
2) Use a good approximation to volumetric center as the test point
3) Also prune the polytope P i.e. drop some of the planes from time to time so $m = O(n)$

F(ω) increases by a fixed constant δ at each step & after K steps

$$\boxed{\text{volume}(P) \leq \left(\frac{n}{\delta}\right)^n e^{-k\delta}}$$

Variants of the algorithm

Desirable properties

(a) Computation at a step as simple as possible
Preferably a single linear system solve

(b) Exploit underlying structure of constraints defining S
eg. Constraints defining S may be explicitly given & each constraint depends only on a few variables.

(c) Polynomial convergence still maintained in the worst case

Possible directions for variants

1) Interpreting the volumetric center as a weighted analytic center and Dynamically weighting the planes
2) Combination of determinant barrier & logarithmic barrier
3) Combination of determinant barriers
4) Several mildly non-linear functions together with a few highly non-linear functions

Several mildly non-linear fns. together with a few highly non-linear ones

$$\max \; p^T x$$

$$\text{s.t. } g_i(x) \geq 0, \quad 1 \leq i \leq m$$

Most of g_i's are only mildly non-linear, g_i's are concave.

$$\phi(\beta, x) = m \ln(p^T x - \beta) + \sum_{i=1}^{m} \ln(g_i(x))$$

Related centering problem

Compute maximizer of $\phi(\beta, x)$

"Lazy use of separating tangent planes"

Centering problem

maximize $\phi(\beta, x)$ where

$$\phi(\beta, x) = m \ln(p^T x - \beta) + \sum_{i=1}^{m} \ln(g_i(x))$$

Alternate between Newton's method & a method that is based on separating (tangent) planes; the subroutine based on separating planes is called only when Newton's method fails to make progress in a consecutive number of steps.

Applications to linear programming

1) The basic algorithm or any suitable variant can solve a linear program with exponentially many constraints as long as there is a good subroutine to generate violated constraints.

Examples — LP relaxations of TSP & maximum independent set.

Weighted matching

2) Possible dynamic weighting of planes in ordinary linear programming.

Size of an s-intersection family in a semilattice and construction of vector space designs by quadratic forms

Prof. Dijen K. Ray-Chaudhuri
Department of Mathematics
Ohio State University

Size of an s-intersection family in a polynomial semilattice and construction of vector-space designs by quadratic forms.

by

D. K. RAY-CHAUDHURI
Ohio State University.

1975 R.M. Wilson and D.K. R-C proved the following: v, k, s, $v \geq k+s$

$|X| = v$ $P_k(X) = \{A : A \subseteq X, |A| = k\}$

Let $\mathcal{A} \subseteq P_k(X)$ such that $|\{|A \cap B| : A \neq B \in \mathcal{A}\}| = s$

Then $|\mathcal{A}| \leq \binom{v}{s}$

T. Zhu and D.K.R-C generalized this result to Polynomial semilattices

Defn. Let $(X, \leq)$ be a partially ordered set. Semi lattice iff for all $x, y \in X$, $x \wedge y$ exists. Assume that the poset has a length function ℓ.

Let $X_i = \{x : \ell(x) = i\}$ $X_0 = \{0\}$ $X = \bigcup_{i=0}^{n} X_i$

Polynomial semilattice iff there exist

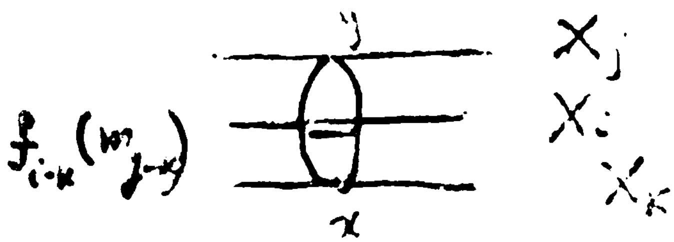
integers $m_0, m_1, \dots, m_n$ and polynom. $f_0, f_1, \cdots f_n$ satisfying

(a) $m_0 < m_1 < \cdots < m_n$

(b) $\deg f_i = i$ and for $i < j$, $f_i \mid f_j$ $\quad i, j = 0, 1, \dots, n$

(c) for all $i, j, k = 0, 1, 2, \dots n$, $k \le i, k \le j$

$$x \in X_k, \; y \in X_j, \; \left|\{ \underset{z \in X_i}{z : x \le z \le y} \}\right| = f_{i-k}(m_{j-k})$$



Ex 1. lattice of subsets $|V| = v$, $X = P(V)$

$X_i = \{ A : A \subseteq V, |A| = i \}$, $i = 0, 1, \dots, v$

$m_i = i \qquad f_i(x) = \binom{x}{i}$

2. Lattice of subspaces, V a vectorspace over a finite field of order q

X_i = subspaces of dim i

$$m_i = q^i \qquad f_i(x) = \frac{(x - q^0)(x - q^1) \cdots (x - q^{i-1})}{(q^i - 1)(q^i - q) \cdots (q^i - q^{i-1})}$$

. Hamming Scheme (orthogonal Array)

$|W| = w$, n a positive integers

$X_i = \{(L,f) : L \subseteq \{1,2,\dots,n\},\ f: L \longrightarrow W,\ |L| = i\}$
$,\ i = 0, 1, \dots, n$

$X = \cup X_i$ $(L_1, f_1) \le (L_2, f_2)$ iff

$L_1 \subseteq L_2$ and $f_2 | L_1 = f_1$.

$m_i = i$ $f_i(x) = \binom{x}{i}$

4. Ordered design, same as 3

with the condition that f is injective

5. q- analogue of Hamming Scheme.

V an n-dim vector space over $GF(q)$

W a w-dim . . . .

$X_i = \{(U,f) : U$ i dim subspace of V
$f: U \longrightarrow W$ linear map$\}$

$(U,f) \le (U', f')$ iff $U \subseteq U'$, $f'|_U = f$

6. q-analogue of ordered design.

Let $(X, \leq)$ be a semilattice with length function ℓ. Let $|x| = \ell(x)$. s be an integer.

$Y \subseteq X$ is called an s-intersection family $|\{ |y \wedge y'|, y \neq y' \in Y\}| = s$.

Thm 1 Let $(X, \leq)$ be a poly. semilattice, s an integer, $Y \subseteq X$ an s-intersection family then $|Y| \leq |X_0| + |X_1| + \cdots + |X_s|$

Thm 2 Assume conditions of Thm 1. Let $Y \subseteq X_k$, k an integer then $|Y| \leq |X_s|$

Thm 3 poly semilattice $(X, \leq)$ $Y \subseteq X_{n_1} \cup X_{n_2} \cup \cdots \cup X_{n_t}$, $n_i \geq s - t + 1$ then $|Y| \leq |X_s| + |X_{s-1}| + \cdots + |X_{s-t+1}|$

Sketch of the proof.

For any poly g, define a matrix $A(Y,g)$ $|Y|\times|Y|$ (y,y') th entry $g(|y\wedge y'|)$

$I(Y,X_i)$ $|Y|\times|X_i|$ matrix

(y,x)th entry $= 1$ if $y \geq x$
0, otherwise

Then $I(Y,X_i)\, I(Y,X_i)^T = A(Y,f_i)$

(y,y') entry $= |\{x : x \leq y \wedge y'\}| = f_i(|y\wedge y'|)$

Columns of $A(Y,f_i)$ are lin comb. of columns of $I[Y,X_i]$.

For a poly g of deg s, col. of $A(Y,g)$ are lin comb of cols $I[Y, X_0 \cup X_1 \cup \cdots \cup X_s]$

Then we find a poly g for which $A[Y,g]$ has rank $|Y|$. $\therefore$

$|Y| \leq |X_0|+|X_1|+\cdots+|X_s|$.

To prove thm 3, we need to show that

rank $I[Y, X_0 \cup X_1 \cup \cdots \cup X_p] =$ rank $I[Y, X_p \cup \cdots \cup X_{p-t+1}]$

i.e columns $X_0, X_1, \ldots X_{p-t}$ are redundant.

Vectorspace Designs

V a v-dim vector space over F_2

X_i be the set of i-dim subspaces.

Let t, k, λ be integers

$\mathcal{B} \subseteq X_k$ is called a t-$[v,k,\lambda;q]$ design if for all $T \in X_t$

$$|\{B : B \in \mathcal{B},\ B \supseteq T\}| = \lambda$$

If repeated blocks are allowed, then we take $\mathcal{B}$ to be a family

$\mathcal{B} = (B_i : i \in I)$ where each $B_i \in X_k$ and ($B_i = B_{i'}$ is possible)

<u>Standard Results</u>

$$|\mathcal{B}| = b = \frac{\lambda \begin{bmatrix} v \\ t \end{bmatrix}}{\begin{bmatrix} k \\ t \end{bmatrix}} \quad \text{where } \begin{bmatrix} m \\ l \end{bmatrix}$$

is the number of l-dim subspaces of an m-dim vector space over F_2.

Let I be a fixed i-dim subspace

Then let $\mathcal{B}_I = \{B : B \in \mathcal{B}, B \supseteq I\}$.

then $b_i = |\mathcal{B}_I| = \dfrac{\lambda \begin{bmatrix} v-i \\ t-i \end{bmatrix}}{\begin{bmatrix} k-i \\ t-i \end{bmatrix}}$

So we get some nec. conditions.

<u>Fisher's</u> <u>Inequality</u> in a $2s$-$[v,k,\lambda,2]$ design $\quad b \geq \begin{bmatrix} v \\ s \end{bmatrix}$ $\quad (v \geq k+s)$

$>$ ineq. holds by a result of L. Chihara.

Analog of Kreher's Result

Let $(V, \mathcal{B})$ be a t-$[v,k,\lambda,q]$ design. $t = 2s$

$G \leq GL(v,q)$ be an auto. group of the design

then $|\mathcal{B}/G| \geq |X_s/G|$

# of block orbits

Some constructions by quadratic forms.

Simon Thomas (1987 Geo. Ded.)

$2-[v,3,7;2]$ for $(v,6)=1$

Let F be a field of order 2^v

F^* = nonzero elements

F a v-dim vector space over $GF(2)$

for $\theta \in F^*$, $x \mapsto \theta x$ a lin. trans

A triple $\{c_1, c_2, c_3\}$ of elements of F is called a special triangle if the pairs $\{c_1, c_2\}$, $\{c_2, c_3\}$, $\{c_3, c_1\}$ belong to the same orbit under F^*

Let $\mathcal{B} = \{\langle c_1, c_2, c_3\rangle \mid \{c_1, c_2, c_3\}$ is a special triangle. Then $\mathcal{B}$ is a $2-[v,3,7;2]$ design which is simple

and non-trivial if $(v, 6) = 1$

E. Schram and myself gen. this

Let F be a field of order q

V an extension of deg v over F

quadratic form

$$Q : V^k \longrightarrow V$$

$$Q(x_1, \dots, x_k) = \sum_{i,j=1}^{k} \alpha_{ij} x_i x_j \quad , \alpha_{ij} \in F$$

$\mathcal{Q}_k$ be the set of non deg. quad. forms. For integers k and j

Let $\mathcal{D}_k^j = \{ (Q, \underline{a}) : Q \in \mathcal{Q}_k$ and $\dim \langle a_1, a_2, \dots, a_k \rangle$

$Q(\underline{a}) = 0$

over $F = j \}$

Let $\mathcal{F}_t = (\mathcal{D}_{t+1}^{t-1} \times I_1) \cup (\mathcal{D}_{t+1}^{t} \times I_{2t} \}$

$\mathcal{I}_t$ is the indexing set of blocks

For $((Q, \underline{a}), i) \in \mathcal{I}_t$, define

$$B((Q, \underline{a}), i) = \langle a_1, a_2, \ldots a_k \rangle_F$$

Let $\mathcal{B}_t = (B(Q, a, i) \mid (Q, a, i) \in \mathcal{I}_t)$

Thm 1 Let v be odd. Then

$(V, \mathcal{B}_t)$ is a $t-[v, K = \{t, t+1\}, \lambda_t, F]$ design. Here λ can be computed.

Define $(Q, \underline{a}, i) \sim (Q', \underline{a}', i')$ if $i = i'$, $\exists\, c \in F^*$ and $R \in GL(k, F)$ such that $Q' = c Q R^{-1}$, $\underline{a}' = R\underline{a}$.

Pick one represent. from each class. Let $\mathcal{B}_t$ be the set of dist

equiv. classes. Then

Thm 2 $(V, \overline{\beta}_t)$ is a $t-[v, k, \overline{\lambda}_t, q]$ design then $\overline{\lambda}_t = \dfrac{|\mathcal{O}_{t-1}|}{q^t (q-1)^2}$

$\overline{\lambda}_2 = q^2+q+1$ $\overline{\lambda}_3 = q^3(q^2+q+1)$

$\overline{\lambda}_4 = q^2(q^2+q+1)(q^3-1)$ - -

For $t=2$, we get Lucky $\mathcal{O}_3^2$ is empty so we get

Thm $(V, \overline{\beta}_2)$ is a $2-[v, 3, q^2+q+1, q]$ - design.

and if $3 \nmid v$, this design is also simple. For $q=2$, we get back Simon Thomas's result.

<u>Thm</u> Let v be odd.

① $\mathcal{B}(\mathcal{D}_k^k)$ is a 2-$[v, k, \lambda^{(k)}, q]$ design with $\lambda^{(k)} = \dfrac{(q^k-1)(q^k-q)}{(q^v-1)(q^v-q)} \, n_k$, $n_k = |\mathcal{Q}_k|$.

② If we take one representative from each equiv. class we get $\overline{\mathcal{B}}(\mathcal{D}_k^k)$ is a 2-$[v, k, \overline{\lambda}_k; q]$ design where $\overline{\lambda}_k = \dfrac{\lambda_k}{(q-1)|\mathcal{Q}_k|}$.

3-design

Let $B \in X_k$. Inflation of B, written as $I(B)$ = the set of $k+1$ dim subsp containing B.

For a family $\mathcal{B} = (B_i : i \in I)$

$I(\mathcal{B})$ = the multiset $\bigcup_{i \in I} I(B_i)$

Thm If $(V, \mathcal{B})$ is a t-$[v, k, \lambda; q]$ design then $I(\mathcal{B})$ is a t-$[v, k+1, \lambda'; q]$ design.

Thm v odd, k even, $k \geq 4$

$\mathcal{B} = \mathcal{B}(\bar{\Delta}_k^k \times I_{q^{\frac{v-k-2}{q-1}}}) \cup I(\mathcal{B}(\Delta_{k-1}^{k-1}$

is a 3-$[v, k, \lambda, q]$ design where λ comb.

Size of an s-intersection family in a polynomial semilattice and construction of vector-space designs by quadratic forms. 1

by

D.K. RAY-CHAUDHURI
Ohio State University

1975 R.M. Wilson and D.K. R-C proved the following: v, k, s, $v \geq k+s$

$|X| = v$ $P_k(X) = \{A : A \subseteq X, |A| = k\}$

Let $\mathcal{C} \subseteq P_k(X)$ such that $|\{|A \cap B| : A \neq B \in \mathcal{C}\}| = s$.

Then $|\mathcal{C}| \leq \binom{v}{s}$

T. Zhu and D.K.R-C generalized this result to Polynomial semilattices

Defn. Let $(X, \leq)$ be a partially ordered set. Semilattice iff for all $x, y \in X$, $x \wedge y$ exists. Assume that the poset has a Length function ℓ.

Let $X_i = \{x : \ell(x) = i\}$ $X_0 = \{0\}$ $X = \bigcup_{i=0}^{n} X_i$

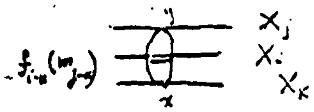
Polynomial semilattice iff there exist integers $m_0, m_1, \ldots, m_n$ and polynom. $f_0, f_1, \cdots f_n$ satisfying 2

(a) $m_0 < m_1 < \cdots < m_n$

(b) $\deg f_i = i$ and for $i < j$, $f_i | f_j$, $i, j = 0, 1, \ldots, n$

(c) for all $i, j, k = 0, 1, 2, \ldots n$, $k \leq i, k \leq j$

$x \in X_k$, $y \in X_j$, $|\{z : x \leq z \leq y, z \in X_i\}| = f_{i-k}(m_{j-k})$



Ex 1. Lattice of subsets $|V| = v$, $X = P(V)$

$X_i = \{A : A \subseteq V, |A| = i\}$, $i = 0, 1, \ldots, v$

$m_i = i$ $f_i(x) = \binom{x}{i}$

2. Lattice of subspaces, V a vectorspace over a finite field of order q

X_i = subspaces of dim i

$m_i = q^i$ $f_i(x) = \dfrac{(x - q^0)(x - q^1) \cdots (x - q^{i-1})}{(q^i - 1)(q^i - q) \cdots (q^i - q^{i-1})}$

3. Hamming Scheme (orthogonal Array) 3

$|W| = w$ n a positive integers

$X_i = \{(L, f) : L \subseteq \{1, 2, \ldots, n\}, f : L \to W, |L| = i\}$, $i = 0, 1, \ldots, n$

$X = \bigcup X_i$ $(L_1, f_1) \leq (L_2, f_2)$ iff

$L_1 \subseteq L_2$ and $f_2 | L_1 = f_1$.

$m_i = i$ $f_i(x) = \binom{x}{i}$

4. Ordered design, same as 3 with the condition that f is injective.

5. q-analogue of Hamming Scheme.

V an n-dim vector space over $GF(q)$

W a w-dim . . .

$X_i = \{(U, f) : U$ i dim subspace of V, $f : U \to W$ linear map$\}$

$(U, f) \leq (U', f')$ iff $U \subseteq U'$, $f'|_U = f$

6. q-analogue of ordered design.

Let $(X, \leq)$ be a semilattice with Length function ℓ. Let $|x| = \ell(x)$. 4

s be an integer.

$Y \subseteq X$ is called an s-intersection family $|\{|y \wedge y'|, y \neq y' \in Y\}| = s$

Thm 1 Let $(X, \leq)$ be a poly. semilattice, s an integer, $Y \subseteq X$ an s-intersection family then $|Y| \leq |X_0| + |X_1| + \cdots + |X_s|$

Thm 2 Assume conditions of Thm 1 Let $Y \subseteq X_k$, k an integer then $|Y| \leq |X_s|$

Thm 3 poly semilattice $(X, \leq)$

$Y \subseteq X_{n_1} \cup X_{n_2} \cup \cdots \cup X_{n_t}$, $n_i \geq s - t + 1$

then $|Y| \leq |X_s| + |X_{s-1}| + \cdots + |X_{s-t+1}|$

Sketch of the proof. 5

For any poly g, define a matrix $A(Y,g)$ $|Y|\times|Y|$ (y,y') th entry $g(|y\wedge y'|)$

$I(Y,X_i)$ $|Y|\times|X_i|$ matrix

(y,x)th entry $= 1$ if $y \geq x$
0, otherwise

Then $I(Y,X_i)\, I(Y,X_i)^T = A(Y,f_i)$

(y,y') entry $= |\{x : x \leq y\wedge y'\}| = f_i(|y\wedge y'|)$

Columns of $A(Y,f_i)$ are lin comb. of columns of $I[Y,X_i]$.

For a poly g of deg s, col. of $A(Y,g)$ are lin comb of cols $I[Y, X_0 \cup X_1 \cup \cdots \cup X_s]$

Then we find poly g for which $A[Y,g]$ has rank $|Y|$.

$|Y| \leq |X_0| + |X_1| + \cdots + |X_s|$.

To prove thm 3, we need to show that 6.

rank $I[Y, X_0 \cup X_1 \cup \cdots \cup X_s]$ = rank $I[Y, X_s \cup \cdots \cup X_{s-t}]$

i.e columns $X_0, X_1, \cdots X_{s-t}$ are redundant.

Vectorspace Designs

V a v-dim vector space over F_q

X_i be the set of i-dim subspaces.

Let t, k, λ be integers

$\mathcal{B} \subseteq X_k$ is called a t-$[v,k,\lambda;q]$ simple design if for all $T \in X_t$

$|\{B : B \in \mathcal{B},\ B \supseteq T\}| = \lambda$

If repeated blocks are allowed, then we take $\mathcal{B}$ to be a family

$\mathcal{B} = (B_i : i \in I)$ where 7

each $B_i \in X_k$ and ($B_i = B_{i'}$ is possible)

Standard Results

$|\mathcal{B}| = b = \dfrac{\lambda \left[{v \atop t}\right]}{\left[{k \atop t}\right]}$ where $\left[{m \atop l}\right]$ is the number of l dim subspaces of an m-dim vector space over F_q.

Let I be a fixed i-dim subspace

Then let $\mathcal{B}_I = \{B : B \in \mathcal{B}, B \supseteq I\}$.

then $b_i = |\mathcal{B}_I| = \dfrac{\lambda \left[{v-i \atop t-i}\right]}{\left[{k-i \atop t-i}\right]}$

So we get some nec. conditions.

Fisher's Inequality in a $2s$-$[v,k,\lambda,q]$ design $b \geq \left[{v \atop s}\right]$ $(v \geq k+s)$

$\Rightarrow$ ineq. holds by a result of L. Chihara. 8.

Analog of Kreher's Result

Let $(V,\mathcal{B})$ be a t-$[v,k,\lambda,q]$ design. $t = 2s$ $G \in GL(v,q)$ be an auto group of the design

then $|\mathcal{B}/G| \geq |X_s/G|$

# of block orbits

Some constructions by quadratic forms. 9.

Simon Thomas (1987 Geo. Ded.)

$2-[v,3,7;2]$ for $(v,6)=1$

Let F be a field of order 2^v

F^* = nonzero elements

F a v-dim vector space over $GF(2)$

For $\theta \in F^*$, $x \mapsto \theta x$ a lin. trans

A triple $\{c_1,c_2,c_3\}$ of elements of F is called a special triangle if the pairs $\{c_1,c_2\}$, $\{c_2,c_3\}$, $\{c_3,c_1\}$ belong to the same orbit under F^*

Let $\mathcal{B} = \{ \langle c_1,c_2,c_3 \rangle \mid \{c_1,c_2,c_3\}$ is a special triangle. Then $\mathcal{B}$ is a $2-[v,3,7;2]$ design which is simple

and nontrivial if $(v,6)=1$ 10

E. Schram and myself gen. this

Let F be a field of order q

V an extension of deg v over F

quadratic form

$Q : V^k \to V$

$Q(x_1,\dots,x_k) = \sum_{i,j=1}^{k} \alpha_{ij} x_i x_j$, $\alpha_{ij} \in F$

$\mathcal{Q}_k$ be the set of non deg. quad. forms. For integers k and j

Let $\mathcal{D}_k^j = \{(Q, \underline{a}) : Q \in \mathcal{Q}_k$, $Q(\underline{a}) = 0$ and $\dim\langle a_1,a_2,\dots,a_k\rangle$ over $F = j\}$

Let $\mathcal{I}_t = (\mathcal{D}_{t+1}^{t+1} \times I_1) \cup (\mathcal{D}_{t+1}^{t} \times I_{2t}\}$

$\mathcal{I}_t$ is the indexing set of blocks 11

For $((Q,\underline{a}),i) \in \mathcal{I}_t$, define

$B((Q,\underline{a}),i) = \langle a_1,a_2,\dots,a_k \rangle_F$

Let $\mathcal{B}_t = (B(Q,a,i) \mid (Q,a,i) \in \mathcal{I}_t)$

Thm 1 Let v be odd. Then $(V,\mathcal{B}_t)$ is a $t-[v, k=\{t,t+1\}, \lambda; q]$ design. Here λ can be computed.

Define $(Q,\underline{a},i) \sim (Q',\underline{a}',i')$ if $i=i'$, $\exists\, c \in F^*$ and $R \in GL(k,F)$ such that $Q' = c\,Q R^{-1}$, $\underline{a}' = R\underline{a}$.

Pick one represent. from each class. Let $\bar{\mathcal{B}}_t$ be the set of dist

equiv. classes. Then 12

Thm 2 $(V,\bar{\mathcal{B}}_t)$ is a $t-[v,k,\bar{\lambda}_t, q]$ design. Also $\bar{\lambda}_t = \dfrac{|\mathcal{Q}_{t+1}|}{q^t (q-1)^2}$

$\bar{\lambda}_2 = q^2+q+1$, $\bar{\lambda}_3 = q^3(q^2+q+1)$

$\bar{\lambda}_4 = q^2(q^2+q+1)(q^3-1)$ - -

For $t=2$, we get Lucky $\mathcal{D}_3^2$ is empty so we get

Thm $(V,\bar{\mathcal{B}}_2)$ is a $2-[v,3,q^2+q+1, q]$ - design.

and if $3 \nmid v$, this design is also simple. For $q=2$, we get back Simon Thomas's result.

$k \geq 3$

Thm Let v be odd.

① $\mathcal{B}(\mathcal{D}_k^k)$ is a 2-$[v, k, \lambda^{(k)}, q]$ design with $\lambda^{(k)} = \dfrac{(q^k-1)(q^k-q)}{(q^v-1)(q^v-q)} n_k$, $n_k = |\mathcal{Q}_k|$.

② If we take one representative from each equiv. class we get $\mathcal{B}(\bar{\mathcal{D}}_k^k)$ is a 2-$[v, k, \bar{\lambda}_k ; q]$ design where $\bar{\lambda}_k = \dfrac{\lambda_k}{(q-1)|\mathcal{Q}_k|}$.

3-design

Let $B \in X_k$. Inflation of B, written as $I(B)$ = the set of $k+1$ dim. subsp containing B.

For a family $\mathcal{B} = (B_i : i \in I)$

$I(\mathcal{B})$ = the multiset $\bigcup_{i \in I} I(B_i)$

Thm If $(V, \mathcal{B})$ is a t-$[v, k, \lambda, q]$ design then $I(\mathcal{B})$ is a t-$[v, k+1, \lambda', q]$ design.

Thm v odd, k even $k \geq 4$

$\mathcal{B} = \mathcal{B}(\bar{\mathcal{D}}_k^k \times I_{\frac{v-k-2}{q-1}}) \cup I(\mathcal{B}(\bar{\mathcal{D}}_{k-1}^{k-1}))$ is a 3-$[v, k, \lambda, 2]$ design where λ const.

A Graph-theoretic Game and its Application to the k-Server Problem

Prof . Douglas B. West
Department of Mathematics
University of Illinois-Urbana

A GRAPH-THEORETIC GAME AND ITS APPLICATION TO THE K-SERVER PROBLEM

Noga Alon

Richard Karp

Douglas West

he Game

Given (G, w), a connected multigraph G
with positive edge weights $w(e)$

Define a matrix game: tree player picks T
edge player picks e

payoff to
edge player is $c(T,e) = \begin{cases} 0 & \text{if } e \in T \\ \dfrac{\text{cycle}(e)}{w(e)} & \text{if } c \notin T \end{cases}$

$$\begin{array}{cc} & \cdots p_T \cdots \\ \vdots\, q\; e & \boxed{\quad c(T,e) \quad} \\ \vdots & \end{array}$$

Mixed strategies:

p = prob. dist on trees (to limit expected loss)

q = prob. dist on edges (to guarantee expected gain)

Minimax Theorem of Game Theory:

$$\underbrace{\min_p}_{\text{optimal for trees}} \max_q \sum\sum \underbrace{p_T q_e c(T,e)}_{\text{expected payoff}} = \underbrace{\max_q}_{\text{optimal for edges}} \min_p \sum\sum \underbrace{p_T q_e c(T,c)}_{\text{expected payoff}}$$

The common value is $Val(G, w)$.

$Val(G)$ if $w \equiv 1$ (unweighted).

Note: $Val(G,w) \le n$ by using pure strategy MST

amples

Complete graph K_n , $w \equiv 1$.

uniform edge strategy guarantees at least

$$\frac{n-1}{\binom{n}{2}} \cdot 0 + \left[1 - \frac{n-1}{\binom{n}{2}}\right] \cdot 3 = 3 - 6/n$$ against any tree,

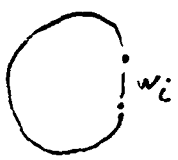
equality only for stars

uniform star tree strategy guarantees at most

$$\frac{2}{n} \cdot 0 + \left[1 - \frac{2}{n}\right] \cdot 3 = 3 - 6/n$$ against any edge.

$\therefore \mathrm{Val}(K_n) = 3 - 6/n$

Weighted cycles



$T_i = C_n - e_i \quad W = \Sigma w_i$

or $p_i = \frac{w_i}{W}$, every edge has expected payoff $(1-\frac{w_i}{W}) \cdot 0 + \frac{w_i}{W} \cdot \frac{W}{w_i} = 1$

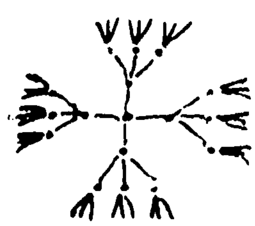
r $q_i = \frac{w_i}{W}$, " tree " " " " " $=1$

$\therefore \mathrm{Val}(C_n, w) = 1$

Cages (unweighted)

$\exists$ 4-regular graphs with girth $c \log n$

$\therefore$ $n+1$ edges of cost $\geq c \log n$ $\quad \mathrm{Val} \in \Omega(\log n)$

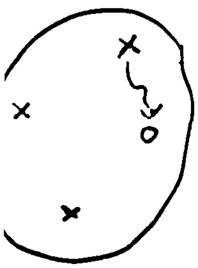


Explicit family: GRIDS!

Conjecture: $\mathrm{Val}(G) \in O(\log n)$?

he k-Server Problem

iven metric space M, service requests processed by k servers.



Process by moving server to request location.

Cost = distance moved by servers.

Initial positions π, request sequence ρ

.et $OPT(\pi,\rho)$ = optimal <u>off-line</u> service cost.

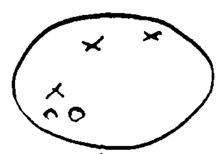
et $A(\pi,\rho)$ = service cost by (deterministic) <u>on-line</u> algorithm A

An on-line algorithm A is c-competitive if
$$A(\pi,\rho) \le c \cdot OPT(\pi,\rho) + a \text{ for all } (\pi,\rho).$$

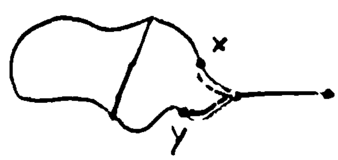
- If $|M| > k$ and $c < k$, $\nexists$ c-competitive deterministic on-line alg.

- Bounded competitiveness always achieveable

  Note: greedy doesn't work



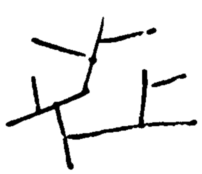
oad network:



$d(x,y)$ = shortest journey

Model by (G,w)

Chrobak-Larmore: For a tree-like road network, there is k-competitive deterministic on-line alg.



ndomized (on-line) algorithm

Algorithm uses outcome of an experiment,
so $A(\pi,\rho)$ is a random variable

Adversary: may specify entire ρ in advance = oblivious
may specify next request based on service choices = adaptive

A is c-competitive if $E(A(\pi,\rho)) \leq c \cdot OPT(\pi,\rho) + a$ for all (π,ρ)

heorem 1: If (G,w) models a road network M, then
$\exists$ a $k(1+Val(G,w))$-competitive randomized on-line
algorithm for the k-server problem on M against an obliv. adv.

roof: Algorithm:

→ Use optimal tree strategy on (G,w) to select tree T.
→ Along each $e \notin T$, pick a random point x_e to cut at.
→ Process ρ along resulting G' using C-L algorithm

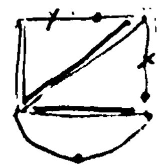
(G,w) → G'

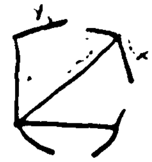
C-L implies $A(\pi,\rho) \leq k\, OPT'(\pi,\rho)$ for all (π,ρ) and experimental outcome G'

suffices to show $E(OPT'(\pi,\rho)) \leq \underline{(1+Val(G,w))}\, OPT(\pi,\rho)$

oof of $E(OPT'(\pi,\rho)) \le [1 + Val(G,w)]\, OPT(\pi,\rho)$

Simulate the moves for $OPT(\pi,\rho)$ on G'.
Ie., when asked to cross roadblock on e
traverse cycle(e) to get to the other side.





Cost of detour is cycle(e) when cross x_e

riven T chosen with probability $p(T)$ . . .

et $d(e)$ = total distance traveled on e by $OPT(\pi,\rho)$

f $e \notin T$, expected # times cross random point x_e is $\frac{d(e)}{w(e)}$

xpected cost of detours for e is $\frac{d(e)}{w(e)} \begin{cases} cycle(e) & \text{if } e \notin T \\ 0 & \text{if } e \in T \end{cases} = d(e)\,c(T,e)$

xpected total cost of <u>detours</u> is

$$\sum_T p(T) \sum_e d(e)\,c(T,e) = OPT(\pi,\rho) \sum_T \sum_e p(T) \left(\frac{d(e)}{OPT}\right) c(T,e)$$

$q(e)$, a distribution

$$\le OPT(\pi,\rho)\ Val(G,w)$$

$\therefore E(OPT'(\pi,\rho)) \le E(\text{this simulation procedure}) \le [1 + Val(G,w)]\, OPT(\pi,\rho)$

Optimization Problem

What is the best tree against the uniform edge strategy?

Let $F_{G,w}(T) = \frac{1}{|E|} \sum_e c(T,e)$

(minimizes the average cost)

Let $v(G,w) = \min_T F_{G,w}(T)$

(use $v(G)$ if $w \equiv 1$)

eorem 2. $\mathrm{Val}(G,w) = \sup_{(G',w')} v(G',w')$, where (G',w') ranges over all weighted multigraphs obtained from (G,w) by replication.

adding copies of ...

oof: Given (G',w'), let $d(e) = \#$ copies of e.

Then $v(G',w') = \min_T \frac{1}{\Sigma d(e)} \sum_e d(e) c(T,e)$

$$= \min_T \sum_e q(e) c(T,e) \quad , \text{ where } \quad q(e) = \frac{d(e)}{\Sigma d(e)}$$

$$\le \max_q \min_T \Sigma q(e) c(T,e) = \max_q \min_p \Sigma\Sigma p_T q_e c(T,e) = \mathrm{Val}(G,w)$$

iven optimal q, (G',w') can approximate it with $d(e) = 1 + \lfloor M q(e) \rfloor$ as $M \to \infty$

eorem 3. If G is edge-transitive, then $\mathrm{Val}(G) = v(G)$.

of: Take T with $F_G(T) = v(G)$ and images $\mathcal{T}$ under $\Gamma(G)$

Play $T' \in \mathcal{T}$ with probability $\frac{1}{|\Gamma(G)|} \#\{\sigma \in \Gamma(G) : \sigma(T) = T'\}$

Then expected payoff for any edge e is

$$\frac{1}{|\Gamma|} \sum_\sigma c(\sigma(T),e) = \frac{1}{|\Gamma|} \sum_\sigma c(T, \sigma^{-1}(e)) \underset{\substack{\uparrow \\ \text{Lagrange}}}{=} \frac{1}{|\Gamma|} \frac{|\Gamma|}{|E(G)|} \Sigma_{e'} c(T,e') = v(G)$$

aximum Val for n-vertex multigraphs - UNWEIGHTED

rem 4

$$\Omega(\log n) \le \underset{\substack{\| \\ f(n)}}{\max_{n(G)=n} \mathrm{Val}(G)} \le e^{c\sqrt{\log n \log\log n}} \qquad \left(\text{or } n^{c\sqrt{\frac{\log n}{\log\log n}}}\right)$$

oof.

→ Suffices to prove that $v(G') \le e^{c\sqrt{\log n \log\log n}}$

→ Begin by reducing attention to multigraphs with $\le n(n+1)$ edges.

Replace G' by H such that $v(G') \le 2v(H)$ and $|E(H)| \le n(n+1)$

H has same underlying graph with D distinct edges as G'.

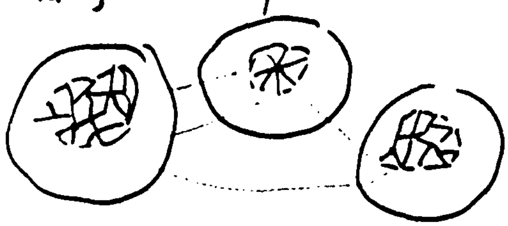
Multiplicities $h(e) = 1 + \left\lfloor \frac{g(e)\,D}{\Sigma g(e)} \right\rfloor$ $\qquad \Sigma h(e) \le 2D$

$h(e) \ge \frac{g(e)D}{\Sigma g(e)}$

$$F_H(T) = \frac{1}{\Sigma h(e)} \Sigma h(e) c(T,e) \ge \frac{1}{2D} \frac{\Sigma g(e) D c(T,e)}{\Sigma g(e)} = \frac{1}{2} F_G(T)$$

→ Recursive construction of tree

Seek large clumps with small diameter and few edges between



Given an integer $x = x(n) > 1$, partition $V(G')$ into parts such that

A) each part has $> x \ln n$ vertices.

B) each part has spanning tree of diameter $< 8x \ln n$.

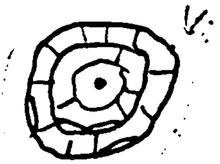
C) fraction of the edges joining vertices in distinct parts $\le \frac{1}{x}$.

Use these trees within these parts, contract parts, and build tree recursively on edges between parts

build partition: Build parts one by one

Components of remaining graph have > $x \ln n$ vertices.

Take a vertex in a remaining component K, stratify by levels.



V_i = vertices at distance i in K (from start).

E_i = edges within V_i or to V_{i-1}

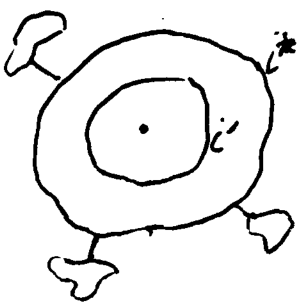
Let i^* = least # shells such that $|V_0 \cup \cdots \cup V_{i^*}| > x \ln n$

and $|E_{i^*+1}| \le \frac{1}{x} |E_1 \cup \cdots \cup E_{i^*}|$.

The new part be $V_1 \cup \cdots V_{i^*}$ and vertices of $K - V_1 \cup \cdots \cup V_{i^*}$

in components of size at most $x \ln n$.

B, C hold by construction. To show diameter < $8x \ln n$:



Let i' = least level so $|V_0 \cup \cdots \cup V_{i'}| > x \ln n$

Note $i' \le x \ln n$ and $|E_1 \cup \cdots \cup E_{i'}| \ge x \ln n$

Claim: $i^* < 3x \ln n$.

Else $|E_1 \cup \cdots \cup E_{i^*}| \ge x \ln n \left(1 + \frac{1}{x}\right)^{i^* - \lfloor x \ln n \rfloor} \ge x \ln n \left(1 + \frac{1}{x}\right)^{2x \ln n}$

$> x \ln n \; n^{2x} > n(n+1)$.

Recurrence: Let $z = 8x(n) \ln n$.

$$f(n) \le \underset{1}{2}\Big[\underset{2}{z} + \underset{3}{\tfrac{1}{x}}\, f\Big(\underset{4}{\tfrac{8n}{z}}\Big)\, \underset{5}{(1+z)}\Big]$$

1 - H instead of G

2 - from diameter bound on parts

3 - fraction of edges between parts

4 - bound on # parts

5 - dilation for passing through parts

With $M = 17 \ln n$, have $f(n) \le M\left[x(n) + f\left(\frac{n}{x(n)}\right)\right]$

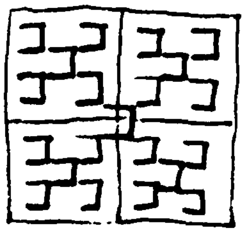
Iterate recurrence, choosing $n_0 = n$ $\quad n_{i+1} = n_i / x(n_i)$.

With $x = c^{c\sqrt{\ln n \ln \ln n}}$, obtain $f(n) \le e^{c'\sqrt{\ln n \ln \ln n}}$

ʒrids (and hypercubes)

heorem 5: For grid G with $N = n^2$ vertices, $v(G) \in \Theta(\lg N)$, and hence $Val(G) \in \Omega(\lg N)$

Jpper bound: Let $n = 2^k$



Define tree T_k by four copies of T_{k-1}, plus center

Diameter $d_k \leq 3 + 2 d_{k-1}$, solution $d_k \leq 3(2^k - 1)$
$= 3(n-1)$

Average cost:

$$F(T_k) \leq \frac{4\left[2 \frac{n}{2}(\frac{n}{2}-1)\right] F(T_{k-1}) + (2n-3)\, d_k}{2n(n-1)} < F(T_{k-1}) + 3 = F(T_0) + 3k = 3 \lg n$$

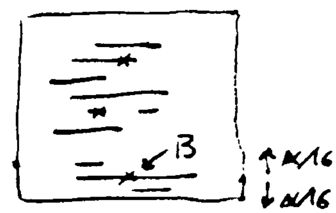
-ower bound: Main idea – show that for an arbitrary tree, some edges yield long cycles, somewhat more yield cycles with a smaller lower bound on length, etc. Count up lower bounds on #edges with given lower bound on cycle(e) and pray!

emma 1: If A is vertex subset with $|A| = \alpha^2 \leq n^2/2$, then $\exists$ at least α rows or at least α columns that A meets but doesn't fill.

·oof: Suppose A hits r rows, s cols, $r \geq s$. Then $rs \geq \alpha^2 \Rightarrow r \geq \alpha$.
Done unless A fills a row, but then $s = n = r$.
If A fills more than $n - \alpha$ rows and $n - \alpha$ columns, then A has more than $n^2 - \alpha^2$ vertices

emma 2 If $|A| = \alpha^2 < n^2/2$ and $|B| \le 4$, then at least $\alpha/2$ vertices of A have neighbors outside A and distance $\ge \alpha/16$ from all of B.

roof: Pick α vertices from distinct rows with outside nbrs; B eliminates $\le \alpha/2$



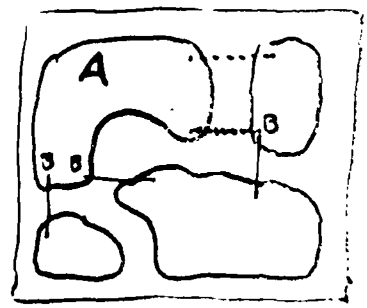
emma 3 For any sp. tree T and $\alpha \le n/4$, at least $n^2/32\alpha$ edges e have cycle$(e) > \alpha/16$

roof: Max degree 4 guarantees bifurcation of any subtree as balanced as $\frac{1}{4}, \frac{3}{4}$.
Iteratively cut biggest till get $m = \lceil n^2/4\alpha^2 \rceil$ pieces.

Claim: smallest piece has $\ge \alpha^2$ vertices. Minimizing x_1 s.t. $x_1 \le \cdots \le x_m \le 4x_1$ and $\sum x_i = M$ sets $x_1 = M/(4m-3)$

Average # deleted edges incident with a piece is ~ 2.
$\therefore$ At least half the pieces incident to at most 4 deleted edges.
Lemma 2 guarantees $\alpha/2$ verts w distance $\ge \alpha/16$ to exit.

$$\left(\ge \frac{1}{2}\ \frac{\text{edges}}{\text{endpt}}\right) \times \left(\ge \frac{\alpha}{2}\ \frac{\text{endpts}}{\text{piece}}\right) \times \left(\ge \frac{n^2}{4\alpha^2} \text{ pieces}\right) = \left(\ge \frac{n^2}{32\alpha} \text{ edges}\right)$$



roof of Theorem: Given T
Choose edge e at random, set $X = c(T,e)$
Then $F(T) = E(X) = \sum_{k \ge 1} \text{Prob}(X \ge k)$.
If $k \le n/64$, set $\alpha = 16k$.
Then $\text{Prob}(X \ge k) \ge \dfrac{n^2/512k}{2n(n-1)} > \dfrac{1}{1024k}$.

$\therefore F(T) \ge \sum_{k=1}^{n/64} \frac{1}{1024k} \sim \frac{\ln n}{1024}$

A GRAPH-THEORETIC GAME AND ITS APPLICATION TO THE K-SERVER PROBLEM

Noga Alon

Richard Karp

Douglas West

The Game

Given (G, w), a connected multigraph G with positive edge weights $w(e)$

Define a matrix game: tree player picks T
edge player picks e

payoff to edge player is $c(T,e) = \begin{cases} 0 & \text{if } e \in T \\ \dfrac{\text{cycle}(e)}{w(e)} & \text{if } e \notin T \end{cases}$

$$\begin{array}{c|c|} & \cdots p_T \cdots \\ \hline \vdots\ q_e & c(T,e) \\ \hline \end{array}$$

Mixed strategies:
p = prob. dist on trees (to limit expected loss)
q = prob. dist on edges (to guarantee expected gain)

Minimax Theorem of Game Theory:

$$\underset{\substack{\uparrow \\ \text{optimal} \\ \text{for trees}}}{\min_p} \max_q \underset{\substack{\uparrow \\ \text{expected} \\ \text{payoff}}}{\sum\sum p_T q_e c(T,e)} = \underset{\substack{\uparrow \\ \text{optimal} \\ \text{for edges}}}{\max_q} \min_p \underset{\substack{\uparrow \\ \text{expected} \\ \text{payoff}}}{\sum\sum p_T q_e c(T,e)}$$

The common value is $Val(G,w)$.

$Val(G)$ if $w \equiv 1$ (unweighted).

Note: $Val(G,w) \le n$ by using pure strategy MST

Examples

1) Complete graph K_n, $w \equiv 1$.

uniform edge strategy guarantees at least
$\frac{n-1}{\binom{n}{2}} \cdot 0 + [1 - \frac{n-1}{\binom{n}{2}}] \cdot 3 = 3 - 6/n$ against any tree, equality only for stars

uniform star tree strategy guarantees at most
$\frac{2}{n} \cdot 0 + [1 - \frac{2}{n}] \cdot 3 = 3 - 6/n$ against any edge.

$\therefore Val(K_n) = 3 - 6/n$ small diameter

2) Weighted cycles

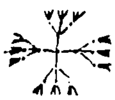
$T_i = C_n - e_i$ $W = \sum w_i$

For $p_i = \frac{w_i}{W}$, every edge has expected payoff $(1-\frac{w_i}{W}) \cdot 0 + \frac{w_i}{W} \cdot \frac{W}{w_i} = 1$
For $q_i = \frac{w_i}{W}$, " tree " " " " " $= 1$

$\therefore Val(C_n, w) = 1$

3) Cages (unweighted)

$\exists$ 4-regular graphs with girth $c \log n$
$\therefore$ all edges of cost $\ge c \log n$ $Val \in \Omega(\log n)$

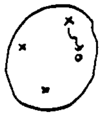


Explicit family: GRIDS!

Conjecture: $Val(G) \in O(\log n)$?

The k-Server Problem

Given metric space M, service requests processed by k servers.



Process by moving server to request location.
Cost = distance moved by servers.
Initial positions π, request sequence ρ

Let $OPT(\pi, \rho)$ = optimal off-line service cost.
Let $A(\pi, \rho)$ = service cost by (deterministic) on-line algorithm A

An on-line algorithm A is c-competitive if
$A(\pi, \rho) \le c \cdot OPT(\pi, \rho) + a$ for all (π, ρ).

→ If $|M| > k$ and $c < k$, $\nexists$ c-competitive deterministic on-line alg.

→ Bounded competitiveness always achieveable
Note: greedy doesn't work

Road network:

$d(x,y)$ = shortest journey

Model by (G, w)

→ Chrobak-Larmore: For a tree-like road network there is k-competitive deterministic on-line alg

Randomized (on-line) algorithm

Algorithm uses outcome of an experiment, so $A(\pi,\rho)$ is a random variable

Adversary: may specify entire ρ in advance = oblivious

may specify next request based on service choices = adaptive

A is c-competitive, if $E(A(\pi,\rho)) \le c \cdot OPT(\pi,\rho) + a$ for all (π,ρ)

against oblivious adv.

Theorem 1: If (G,w) models a road network M, then $\exists$ a $k(1+Val(G,w))$-competitive randomized on-line algorithm for the k-server problem on M against an oblix. adv.

Proof: Algorithm:

→ Use optimal tree strategy on (G,w) to select tree T.

→ Along each $e \notin T$, pick a random point x_e to cut at.

→ Process ρ along resulting G' using C-L algorithm

C-L implies $A(\pi,\rho) \le k\ OPT'(\pi,\rho)$ for all (π,ρ) and experimental outcome G'

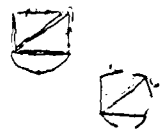
suffices to show $E(OPT'(\pi,\rho)) \le \underline{(1+Val(G,w))}\ OPT(\pi,\rho)$

no on-line algorithm involved!

Proof of $E(OPT'(\pi,\rho)) \le [1+Val(G,w)]\ OPT(\pi,\rho)$

Simulate the moves for $OPT(\pi,\rho)$ on G'.

I.e., when asked to cross roadblock on e traverse cycle(e) to get to the other side.



Cost of detour is cycle(e) when cross x_e

→ Given T chosen with probability $p(T)$ . .

Let $d(e)$ = total distance traveled on e by $OPT(\pi,\rho)$

If $e \notin T$, expected # times cross random point x_e is $\frac{d(e)}{w(e)}$

Expected cost of detours for e is $\frac{d(e)}{w(e)}\begin{Bmatrix} cycle(e) & if\ e\notin T \\ 0 & if\ e\in T\end{Bmatrix} = d(e)c(T,e)$

Expected total cost of <u>detours</u> is

$\sum_T p(T) \sum_e d(e)c(T,e) = OPT(\pi,\rho) \sum_T \sum_e p(T)\left(\frac{d(e)}{OPT}\right)c(T,e)$

$\le OPT(\pi,\rho)\ Val(G,w)$ $\qquad q(e)$, a distribution

$\therefore E(OPT'(\pi,\rho)) \le E(\text{this simulation procedure}) \le [1+Val(G,w)]\ OPT(\pi,\rho)$

An Optimization Problem

. What is the best tree against the uniform edge strategy?

Let $F_{G,w}(T) = \frac{1}{|E|}\sum_e c(T,e)$

(minimizes the average cost)

Let $v(G,w) = \min_T F_{G,w}(T)$

(use $v(G)$ if $w \equiv 1$)

Theorem 2. $Val(G,w) = \sup_{(G',w')} v(G',w')$, where (G',w') ranges over all weighted multigraphs obtained from (G,w) by replication.

creating copies of edges with same weight

Proof: Given (G',w'), let $d(e)$ = # copies of e.

Then $v(G',w') = \min_T \frac{1}{\sum d(e)}\sum_e d(e)c(T,e)$

$= \min_T \sum_e q(e)c(T,e)$, where $q(e) = \frac{d(e)}{\sum d(e)}$

$\le \max_q \min_T \sum q(e)c(T,e) = \max_q \min_p \sum\sum p_T q_e c(T,e) = Val(G,w)$

Given optimal q, (G',w') can approximate it with $d(e) = 1 + \lfloor M q(e) \rfloor$ as $M \to \infty$

Theorem 3. If G is edge-transitive, then $Val(G) = v(G)$.

Proof: Take T with $F_G(T) = v(G)$ and images σT under $\Gamma(G)$

Play $T' = \sigma T$ with probability $\frac{1}{|\Gamma(G)|}\#\{\sigma \in \Gamma(G): \sigma(T) = T'\}$

Then expected payoff for any edge e is

$\frac{1}{|\Gamma|}\sum_\sigma c(\sigma(T),e) = \frac{1}{|\Gamma|}\sum_\sigma c(T,\sigma^{-1}(e)) \underset{\text{Lagrange}}{=} \frac{1}{|\Gamma|}\frac{|\Gamma|}{|E(G)|}\sum_{e'} c(T,e') = v(G)$

Maximum Val for n-vertex multigraphs - UNWEIGHTED

Theorem # $\Omega(\log n) \le \underbrace{\max_{v(G)=n} Val(G)}_{f(n)} \le e^{c\sqrt{\log n \log\log n}}$ (or $n^{c\sqrt{\frac{\log\log n}{\log n}}}$)

Proof.

→ Suffices to prove that $v(G') \le e^{c\sqrt{\log n \log\log n}}$

→ Begin by reducing attention to multigraphs with $\le n(n-1)$ edges.

Replace G' by H such that $v(G') \le 2v(H)$ and $|E(H)| \le n(n-1)$

H has same underlying graph with D distinct edges as G'.

Multiplicities $h(e) = 1 + \left\lfloor \frac{g(e)\,D}{\sum g(e)} \right\rfloor$

$\Sigma h(e) \le 2D$

$h(e) \ge \frac{g(e)D}{\Sigma g(e)}$

$F_H(T) = \frac{1}{\Sigma h(e)}\sum h(e)c(T,e) \ge \frac{1}{2D}\frac{\sum g(e)Dc(T,e)}{\sum g(e)} = \frac{1}{2}F_G(T)$

→ Recursive construction of tree

Seek large clumps with small diameter and few edges between

Given an integer $x = x(n) > 1$, partition $V(G')$ into parts such that

A) each part has $> x \ln n$ vertices

B) each part has spanning tree of diameter $< 8x\ln n$.

C) fraction of the edges joining vertices in distinct parts $\le \frac{1}{x}$

Use these trees within these parts, contract parts, and build tree recursively on edges between parts

To build partition: Build parts one by one

Components of remaining graph have $= x \ln n$ vertices.

Take a vertex in a remaining component K, stratify by levels.

V_i = vertices at distance i in K (from start).

E_i = edges within V_i or to V_{i-1}

Let i^* = least # shells such that $|V_0 \cup \cdots \cup V_{i^*}| > x \ln n$

and $|E_{i^*+1}| \le \frac{1}{x} |E_1 \cup \cdots \cup E_{i^*}|$.

Let the new part be $V_0 \cup \cdots \cup V_{i^*}$ and vertices of $K - V_0 \cup \cdots \cup V_{i^*}$ in components of size at most $x \ln n$.

→ A, C hold by construction. To show diameter $< 8x \ln n$:

Let i' = least level so $|V_0 \cup \cdots \cup V_{i'}| > x \ln n$

Note $i' \le x \ln n$ and $|E_1 \cup \cdots \cup E_{i'}| \ge x \ln n$

Claim: $i^* < 3x \ln n$.

Else $|E_1 \cup \cdots \cup E_{i^*}| \ge x \ln n \, (1+\frac{1}{x})^{i^* - \lfloor x \ln n \rfloor} \ge x \ln n \, (1+\frac{1}{x})^{2x \ln n}$

$> x \ln n \; n^{2n} > n(n-1)$.

→ Recurrence: Let $z = 8x(n) \ln n$.

$$f(n) \le \underset{1}{2}\Big[\underset{2}{z} + \underset{3}{\tfrac{1}{x}}\, \underset{4}{f\big(\tfrac{8n}{z}\big)}\, \underset{5}{(1+z)}\Big]$$

1 = α instead of $\mathscr{C}$
2 = from diameter bound on parts
3 = fraction of edges between parts
4 = bound on # parts
5 = dilation for passing through parts

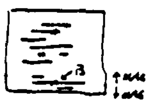
With $M = 17 \ln n$, have $f(n) \le M\big[x(n) + f\big(\frac{n}{x(n)}\big)\big]$

Iterate recurrence, choosing $n_0 = n$ $\quad n_{i+1} = n_i / x(n_i)$.

With $x = e^{c\sqrt{\ln n \ln \ln n}}$, obtain $f(n) \le e^{c'\sqrt{\ln n \ln \ln n}}$

Lemma 2 If $|A| = \alpha^2 < n^2/2$ and $|B| \le 4$, then at least $\alpha/2$ vertices of A have neighbors outside A and distance $\ge \alpha/16$ from all of B.

Proof: Pick α vertices from distinct rows with outside nbrs; B eliminates $\le \alpha/2$



Lemma 3 For any sp. tree T and $\alpha \le n/4$, at least $n^2/32\alpha$ edges e have cycle$(e) > \alpha/16$

Proof: Max degree 4 guarantees bifurcation of any subtree as balanced as $\frac{1}{4}, \frac{3}{4}$.

Iteratively cut biggest till get $m = \lceil n^2/4\alpha^2 \rceil$ pieces.

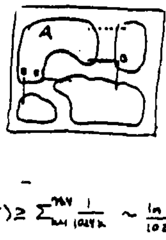
Claim: smallest piece has $\ge \alpha^2$ vertices. (Minimizing x_1 st. $x_1 \le \cdots \le x_m \le 4x_1$ and $\sum x_i = M$ sets $x_1 = M/(4m-3)$)

Average # deleted edges incident with a piece is ~ 2.

∴ At least half the pieces incident to at most 4 deleted edges.

Lemma 2 guarantees $\alpha/2$ verts w distance $\ge \alpha/16$ to exit.

$\left(\ge \frac{1}{2} \frac{\text{edges}}{\text{endpt}}\right) \times \left(\ge \frac{\alpha}{2} \frac{\text{endpts}}{\text{piece}}\right) \times \left(\ge \frac{n^2}{4\alpha^2} \text{ pieces}\right) = \left(\ge \frac{n^2}{32\alpha} \text{ edges}\right)$



Proof of Theorem: Given T

Choose edge e at random, set $X = c(T, e)$ (cycle(e))

Then $F(T) = E(X) = \sum_{k \ge 1} \text{Prob}(X \ge k)$.

If $k \le n/64$, set $\alpha = 16k$.

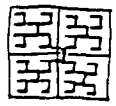
Then $\text{Prob}(X \ge k) \ge \frac{n^2/512k}{2n(n-1)} > \frac{1}{1024k}$.

$\therefore F(T) \ge \sum_{k=1}^{n/64} \frac{1}{1024k} \sim \frac{\ln n}{1024}$

Grids (and hypercubes)

Theorem 5. For grid G with $N = n^2$ vertices, $v(G) \in \Theta(\lg N)$, and hence $\text{Val}(G) \in \Omega(\lg N)$

Upper bound: Let $n = 2^k$



Define tree T_k by four copies of T_{k-1}, plus center

Diameter $d_k \le 3 + 2d_{k-1}$, solution $d_k \le 3(2^k - 1) = 3(n-1)$

Average cost:

$$F(T_k) \le \frac{4\,[2\frac{n}{2}(\frac{n}{2}-1)]\,F(T_{k-1}) + (2n-3)\,d_k}{2n(n-1)} < F(T_{k-1}) + 3 = F(T_0) + 3k = 3 \lg n$$

Lower bound: Main idea – show that for an arbitrary tree, some edges yield long cycles, somewhat more yield cycles with a smaller lower bound on length, etc. Count up lower bounds on # edges with given lower bound on cycle(e) and pray!

Lemma 1: If A is vertex subset with $|A| = \alpha^2 \le n^2/2$, then $\exists$ at least α rows or at least α columns that A meets but doesn't fill.

Proof: Suppose A hits r rows, s cols, $r \ge s$. Then $rs \ge \alpha^2 \Rightarrow r \ge \alpha$.

Done unless A fills a row, but then $s = n = r$.

If A fills more than $n - \alpha$ rows and $n - \alpha$ columns, then A has more than $n^2 - \alpha^2$ vertices

Polynomial-Time Algorithms from Finite Basis Theorems - A Survey

Prof. Michael Langston
Department of Computer Science
University of Tennessee

POLYNOMIAL-TIME ALGORITHMS FROM FINITE BASIS THEOREMS
— A SURVEY —

MIKE LANGSTON

- RELEVANT GRAPH METRICS
- DECISION ALGORITHMS
- SEARCH ALGORITHMS
- CONSTRUCTIVIZATIONS
- APPROXIMATE BASES
- FUTURE WORK

SOME AMENABLE METRICS

CUTWIDTH, MOD. CUTWIDTH

PATHWIDTH, TREEWIDTH

SEARCH NUMBER, NODE SEARCH NUMBER

⋮

NP-COMPLETE IN GENERAL

FOR FIXED WIDTH (k), BEST PREVIOUS BOUNDS: OPEN

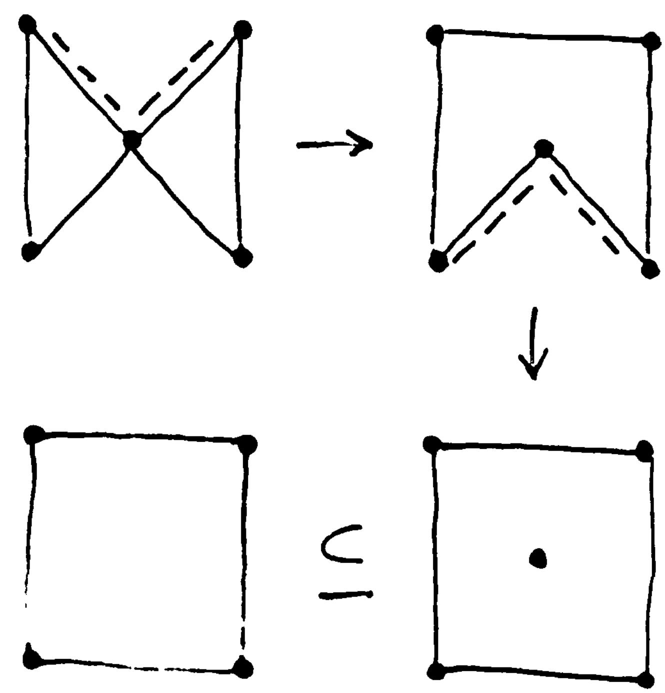
$O(\exp(k))$

NEW BOUNDS: $O(n^2)$

- IMMERSION ORDER ($\leq_i$)
  - TAKE SUBGRAPH
  - LIFT PAIR OF EDGES

- EXAMPLE: $C_4 \leq_i K_1 + 2K_2$ ($\not\leq_t$, $\not\leq_m$)

$K_1 + 2K_2$



C_4

THM [RS] (NASH-WILLIAMS' CONJ)

$\leq_i$ IS WPO

THM [FL] $\exists$ POLY-TIME TEST

FOR $H \leq_i G$ FOR EVERY

FIXED H.

- CLOSURE: $\left.\begin{array}{l} G \in F \\ H \leq_i G \end{array}\right\} \Rightarrow H \in F$

F LOWER IDEAL

- OBSTRUCTION SET

- IMMERSION-CLOSED $\Rightarrow$ POLY-TIME DECISION ALG

- NONCONSTRUCTIVE AT TWO LEVELS

- HORRIBLE CONSTANTS

- TIME COMPLEXITY:

  $\mathcal{O}(|V|^{h+6})$ WHERE h DENOTES ORDER OF LARGEST OBSTRUCTION

SAMPLE APPLICATION:

- k-MIN CUT

  $- \mathcal{O}(|V|^{k-1})$ [MS]

  - NEITHER "YES" NOR "NO" FAMILIES CLOSED $\leq_m$

  - "YES" FAMILY CLOSED $\leq_i$

  $\therefore \in P$

SELF-REDUCIBILITY $\mathcal{E}(|V|^{k+6+\underline{2}})$

OBVIOUS ORACLE CALLS
CAN ↓ TO 1

BETTER BOUNDS FOR $\leq_i$

k-MIN CUT $[O(|V|^{k+6})]$

- "YES" FAMILY EXCLUDES A (LARGE) BINARY TREE, T
- $T \leq_m G \Rightarrow T \leq_i G$

  $\Rightarrow G \in$ "NO" FAMILY
- "YES" FAMILY HAS BOUNDED TREE-WIDTH
- TREE DECOMPOSITION, $\leq_i$ TESTS, ETC ARE $O(|V|^2)$

MA:
DEC
3

"YES
CLOS

$\forall$ solu
that
1) cover
2) cove
3) have
$\cap$ proj

CONSTRUCTIVITY

$(R, \leq)$ IS UNIFORMLY ENUMERABLE IF $\exists$ RECURSIVE ENUMERATION $(r_0, r_1, \ldots)$ OF $R \ni r_i \leq r_j \Rightarrow i \leq j$.

UNDER SUCH AN ENUMERATION, A SELF-RED. ALG IS UNIFORM IF, ON INPUT r_j, IT CONSULTS ITS ORACLE ONLY FOR r_i WITH $i \leq j$.

EXAMPLE: R = FINITE GRAPHS

$\leq_m$ (OR $\leq_i$)

SELF-RED. BY EDGE DELETION

#9

MAIN RESULT ON WHICH ALL ELSE IS BASED

<u>THM</u>. IF F IS CLOSED IN A UNIF. ENUM. WPO $\exists$ WE KNOW ALGS TO:

1) <u>CHECK</u> CANDIDATE SEARCH SOLUTION IN $\underline{O(T_1(n))}$ TIME

2) PERFORM ANY <u>OBSTRUCTION TEST</u> IN $\underline{O(T_2(n))}$ TIME

3) DO <u>UNIFORM SELF-RED</u>

4) DO (SOME FORM OF) SELF-RED WITH $O(T_3(n))$ CALLS & OVERHEAD,

THEN WE CAN CONSTRUCT AN $\underline{O(\max\{T_1(n), T_2(n) \cdot T_3(n)\})}$ TIME ALG TO COMPUTE A SEARCH FCN FOR F (& HENCE DECIDE MEMBERSHIP IN F).

EXAMPLE: k MIN CUT

CHECKING $T_1(n) = \underline{O}(n)$

OBST TESTS $T_2(n) = \underline{O}(n^2)$

UNIF. SELF-RED. (SLOW, USES GADGET)

SELF-RED. $T_3(n) = \underline{O}(n)$ (WE USE SCAFFOLDING & AMORTIZED ANALYSIS)

COROLLARY: k MIN CUT CAN BE SOLVED IN $O(n^3)$ TIME WITH A KNOWN ALGORITHM.

Bod $\downarrow$ n^2

OBSERVATION: $\exists$ A LACK OF SYM. WORKS ONLY FOR "YES" CLOSURE

be in some family [illegible]

PRACTICAL POSSIBILITIES:

- LEARNING SYSTEM
- SMALL # OF REQ'D OBSTS
- FAST OBST TESTS

THEORETICAL EXTENSION:

- RETAIN OBSTRUCTION TESTING & GENERATION (IN CASE "NO")
- REPLACE SELF-RED. WITH TM DIAGONALIZATION (IN CASE "YES")
- INTERLEAVE

SUPPOSE AN NP-COMPLETE PROBLEM IS FOUND TO BE MINOR- OR IMMERSION-CLOSED?

THEN P=NP NONCONSTRUCTIVELY

NOT ONLY A SURPRISE BUT, WHAT'S WORSE, NP-COMPLETENESS ROOFS DO NOT (AS GENERALLY CLAIMED) PROVIDE POLY-TIME ALGS!

(Cannot immediately apply last line (may not be checkable or decidable.)

<u>BUT</u>, AS LONG AS THE POLY-TIME REDUCTION IS KNOWN, WE CAN EXPLOIT:

1) SAT $\propto \Pi$ AND Π CLOSED $\Rightarrow$ IMAGE OF SAT CLOSED

TECHNICAL LEMMA

2) CHECKING SOLUTIONS & UNIFORMLY SELF-REDUCING SAT ARE EASY

YIELDING A CONSTRUCTIVE P=NP PROOF (WITHOUT ANY MEANS FOR ISOLATING THE RELEVANT OBST. SET).

APPROXIMATE OBSTRUCTION SETS

- USE: {HANDY OBSTS} $\cup$ {HANDY NON-OBSTS}

"YES"	$\not\geq$ HANDY NON-OBST		← FALSE NEGATIVES
"NO"	$\geq$ HANDY OBST		← FALSE POSITIVES

- EXAMPLE: 3-GML (2-PW) AND $\{K_4\} \cup \emptyset$ ∃ 110 !

  [FAST K_4 TOPO TEST YIELDS FEW FP's]

  4-GML (3-PW) AND $\{K_5\} \cup \{K_{3,3}\}$

  [FAST PLANARITY TEST YIELDS FEW FP's, FN's]

- MAKES SELF-REDUCTION INTERESTING:

  "YES" INSTANCE → "YES" EVIDENCE WITHOUT FP.

QUESTIONS

- PRACTICALITY: NONCONSTRUCTIVE ✓
  DECIDE ONLY ✓
  HUGE CONSTANTS ~
- BANDWIDTH: TOPO, $\Delta \leq 3$ ✓
  GENERAL, FOR FIXED k ?
- TREE-DECOMP BOTTLENECK: $\mathcal{O}(n^2)$ ✓
  FASTER ?
  ($\mathcal{O}(n \log^2 n)$ RECENTLY CLAIMED)
- PARALLEL ALGS: TREE-DECOMP ✓
  ORDER TESTS ✓
  DISJOINT PATHS ?
- SELF-REDUCTION: ALWAYS POSSIBLE ?
  ALWAYS UNIFORM ?

POLYNOMIAL-TIME ALGORITHMS FROM FINITE BASIS THEOREMS
— A SURVEY —

MIKE LANGSTON

- RELEVANT GRAPH METRICS
- DECISION ALGORITHMS
- SEARCH ALGORITHMS
- CONSTRUCTIVIZATIONS
- APPROXIMATE BASES
- FUTURE WORK

SOME AMENABLE METRICS

CUTWIDTH, MOD. CUTWIDTH

PATHWIDTH, TREEWIDTH

SEARCH NUMBER, NODE SEARCH NUMBER

⋮

NP-COMPLETE IN GENERAL

FOR FIXED WIDTH (k), BEST PREVIOUS BOUNDS: OPEN
$O(\exp(k))$

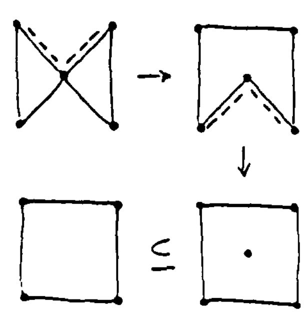
NEW BOUNDS: $O(n^2)$

- IMMERSION ORDER ($\leq_i$)
  - TAKE SUBGRAPH
  - LIFT PAIR OF EDGES

- EXAMPLE: $C_4 \leq_i K_1 + 2K_2$ ($\not\leq_t$, $\not\leq_m$)

$K_1 + 2K_2$

C_4



#4

#3 THM [RS] (NASH-WILLIAMS' CONJ)
$\leq_i$ IS WPO

THM [FL] $\exists$ POLY-TIME TEST FOR $H \leq_i G$ FOR EVERY FIXED H.

- CLOSURE: $\left. \begin{array}{l} G \in F \\ H \leq_i G \end{array} \right\} \Rightarrow H \in F$

- OBSTRUCTION SET

- IMMERSION-CLOSED $\Rightarrow$ POLY-TIME DECISION ALG

5

- NONCONSTRUCTIVE AT TWO LEVELS
- HORRIBLE CONSTANTS
- TIME COMPLEXITY: $O(|V|^{h+6})$ WHERE h DENOTES ORDER OF LARGEST OBSTRUCTION

6

SAMPLE APPLICATION:

- k-MIN CUT
  - $O(|V|^{k-1})$ [MS]
  - NEITHER "YES" NOR "NO" FAMILIES CLOSED $\leq_m$
  - "YES" FAMILY CLOSED $\leq_i$

  $\therefore \in P$
- SELF-REDUCIBILITY $O(|V|^{h+6+2})$

7

BETTER BOUNDS FOR $\leq_i$

k-MIN CUT $[O(|V|^{h+6})]$

- "YES" FAMILY EXCLUDES A (LARGE) BINARY TREE, T
- $T \leq_m G \Rightarrow T \leq_i G$

  $\Rightarrow G \in$ "NO" FAMILY
- "YES" FAMILY HAS BOUNDED TREE-WIDTH
- TREE DECOMPOSITION, $\leq_i$ TESTS, ETC ARE $O(|V|^2)$

8

CONSTRUCTIVITY

- $(R, \leq)$ IS UNIFORMLY ENUMERABLE IF $\exists$ RECURSIVE ENUMERATION $(r_0, r_1, \ldots$ OF $R \ni r_i \leq r_j \Rightarrow i \leq j$.
- UNDER SUCH AN ENUMERATION, A SELF-RED. ALG IS UNIFORM IF, ON INPUT r_j, IT CONSULTS ITS ORACLE ONLY FOR r_i WITH $i \leq j$.

EXAMPLE: R = FINITE GRAPHS
$\leq_m$ (OR $\leq_i$)
SELF-RED. BY EDGE DELETION

THM. IF F IS CLOSED IN A UNIF. ENUM. WPO $\ni$ WE KNOW ALGS TO:

1) CHECK CANDIDATE SEARCH SOLUTION IN $\mathcal{O}(T_1(n))$ TIME
2) PERFORM ANY OBSTRUCTION TEST IN $\mathcal{O}(T_2(n))$ TIME
3) DO UNIFORM SELF-RED
4) DO (SOME FORM OF) SELF-RED WITH $\mathcal{O}(T_3(n))$ CALLS & OVERHEAD,

THEN WE CAN CONSTRUCT AN $\mathcal{O}(\max\{T_1(n), T_2(n)\cdot T_3(n)\})$ TIME ALG TO COMPUTE A SEARCH FCN FOR F (& HENCE DECIDE MEMBERSHIP IN F).

EXAMPLE: k MIN CUT

- CHECKING $T_1(n) = \mathcal{O}(n)$
- OBST TESTS $T_2(n) = \mathcal{O}(n^2)$
- UNIF. SELF-RED. (SLOW, USES GADGET)
- SELF-RED. $T_3(n) = \mathcal{O}(n)$ (WE USE SCAFFOLDING & AMORTIZED ANALYSIS)

COROLLARY: k MIN CUT CAN BE SOLVED IN $\mathcal{O}(n^3)$ TIME WITH A KNOWN ALGORITHM.

OBSERVATION: $\exists$ A LACK OF SYM. WORKS ONLY FOR "YES" CLOSURE

PRACTICAL POSSIBILITIES:

- LEARNING SYSTEM
- SMALL # OF REQ'D OBSTS
- FAST OBST TESTS

THEORETICAL EXTENSION:

- RETAIN OBSTRUCTION TESTING & GENERATION (IN CASE "NO")
- REPLACE SELF-RED. WITH TM DIAGONALIZATION (IN CASE "YES")
- INTERLEAVE

SUPPOSE AN NP-COMPLETE PROBLEM IS FOUND TO BE MINOR- OR IMMERSION-CLOSED?

THEN P = NP NONCONSTRUCTIVELY

NOT ONLY A SURPRISE BUT, WHAT'S WORSE, NP-COMPLETENESS PROOFS DO NOT (AS GENERALLY CLAIMED) PROVIDE POLY-TIME ALGS!

BUT, AS LONG AS THE POLY-TIME REDUCTION IS KNOWN, WE CAN EXPLOIT:

1) SAT $\propto \Pi$ AND Π CLOSED $\Rightarrow$ IMAGE OF SAT CLOSED

2) CHECKING SOLUTIONS & UNIFORMLY SELF-REDUCING SAT ARE EASY

YIELDING A CONSTRUCTIVE P=NP PROOF (WITHOUT ANY MEANS FOR ISOLATING THE RELEVANT OBST. SET).

QUESTIONS

- PRACTICALITY: NONCONSTRUCTIVE ✓
  - DECIDE ONLY ✓
  - HUGE CONSTANTS ~
- BANDWIDTH: TOPO, $\Delta \leq 3$ ✓
  - GENERAL, FOR FIXED k ?
- TREE-DECOMP BOTTLENECK: $O(n^2)$ ✓
  - FASTER ?
  - ($O(n \log^2 n)$ RECENTLY CLAIMED)
- PARALLEL ALGS: TREE-DECOMP ✓
  - ORDER TESTS ✓
  - DISJOINT PATHS ?
- SELF-REDUCTION: ALWAYS POSSIBLE ?
  - ALWAYS UNIFORM ?

APPROXIMATE OBSTRUCTION SETS

- USE: {HANDY OBSTS} ∪ {HANDY NON-OBSTS}

"YES"	$\not\geq$ HANDY NON-OBST		← FALSE NEGATIVES
"NO"	$\geq$ HANDY OBST		← FALSE POSITIVES

- EXAMPLE: 3-GML (2-PW) AND $\{K_4\} \cup$ (? 3/110 !
  [FAST K_4 TOPO TEST YIELDS FEW FP's]
  4-GML (3-PW) AND $\{K_5\} \cup \{K_{3,3}\}$
  [FAST PLANARITY TEST YIELDS FEW FP's, FN's]
- MAKES SELF-REDUCTION INTERESTING:
  "YES" INSTANCE → "YES" EVIDENCE WITHOUT FP.

On the Product of the independent Domination Numbers of a Graph and its Complement

Prof. Gerd H. Fricke
Department of Mathematics and Statistics
Wright State University

On the Product of the Independence Domination Numbers of a Graph and its Complement

Gerd H. Fricke
Wright State U.

For a graph G let

$i(G) = \min \{ |S| : S \text{ is a maximal independent set} \}$

$\quad = \min \{ |S| : S \text{ is a dominating independent set} \}$.

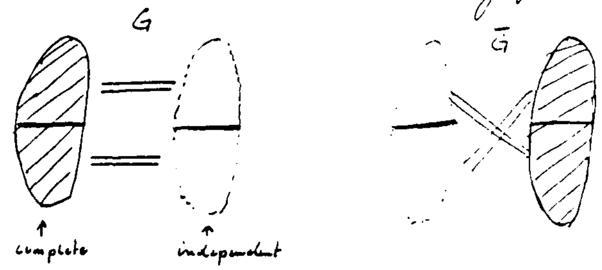
$i(\bar{G})$ is the smallest cardinality of a maximal clique in G.

Let $mii(p) = \max\limits_{|G|=p} \{ i(G)\, i(\bar{G}) \}$.

It is easy to show that $mii(p)$ is nondecreasing.

Also $\left\lfloor \frac{(p+3)^2}{16} \right\rfloor \le mii(p)$

Let $p = 4m$ and consider the following graph G



$i(G) = m+1$ $\qquad$ $i(\bar{G}) = m+1$

$mii(p) \ge \frac{(p+4)^2}{16}$

Now, $i(G) + i(\bar{G}) \le p+1$ and thus

$\left\lfloor \frac{(p+3)^2}{16} \right\rfloor \le mii(p) \le \frac{(p+1)^2}{4}$

Recently Cockayne, Favaron, Li, and MacGillivray showed that

$mii(p) \le \min \left\{ \frac{(p+3)^2}{8}, \frac{(p+8)^2}{10.8} \right\}$.

Theorem. Let $0 < K < 16$ then there exists an integer p_0 such that

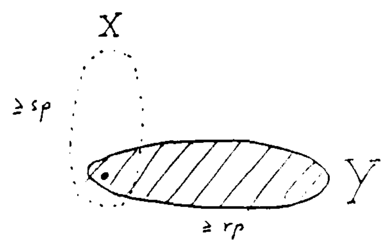
$mii(p) \le \frac{p^2}{K}$ for all $p \ge p_0$.

$\left(\lim\limits_{p \to \infty} \frac{mii(p)}{p^2} = \frac{1}{16} \right)$

Proof: Let $0 < K < 16$ and let G be a graph of p vertices such that $i(G)\, i(\bar{G}) > \frac{p^2}{K}$.

Note that any vertex is contained in an independent set of size $\ge i(G)$ and a clique of size $i(\bar{G})$.

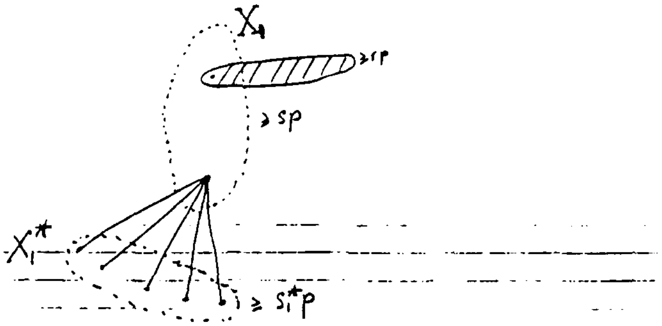
Let $sp = i(G)$ and $rp = i(\bar{G})$ and assume $r \le s$.



X independent
Y complete

Each vertex in X is contained in a complete K_{rp} and is adjacent to at least $rp-1$ vertices in $V \setminus X$. Thus we have at least $sp(rp-1)$ edges to $p - sp$ points in $V \setminus X$.

Let $\frac{sp(rp-1)}{p-sp} = s_1^+ p$ then there exist a point v_1 such that $v \in V \setminus X$ and is adjacent to at least $s_1^+ p$ vertices in X.



Again any vertex $v \in X_1$ is contained in a maximal clique of size $\geq rp$ and that clique contains at most one vertex of X_1 (namely v) and at most one vertex of X_1^+.

Thus every $v \in X_1$ has $rp-2$ edges to vertices not in X_1 or X_1^+.

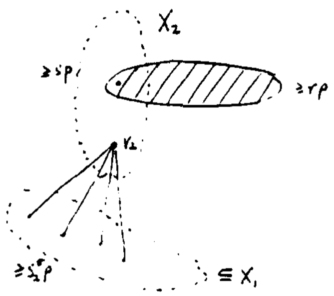
Let $\frac{sp\,(rp-2)}{p - s_1^+p - sp} = s_2^+ p$

Hence there exist a $v_2 \notin X_1 \cup X_1^+$ that is adjacent to $s_2^+ p$ vertices of X_1

Simplify and we have

$$\frac{s\,(r-\frac{2}{p})}{1-s_1^+-s} = s_2^+$$

and repeat the argument.



Let $\frac{s(r-\frac{2}{p})}{1-s_2^+-s} = s_3^+$ and in general

$$\frac{s\,(r-\frac{2}{p})}{1-s_n^+-s} = s_{n+1}^+.$$

$\{s_n^+\}$ is increasing and let $\lim s_n^+ = x$

Then $\frac{s(r-\frac{2}{p})}{1-x-s} = x$ and

$$x^2 - (1-s)\,x + sr - \frac{2s}{p} = 0$$

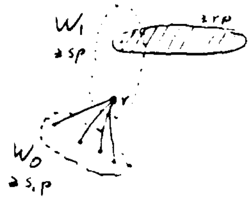
5

$$x = \frac{1-s-\sqrt{(1-s)^2 - 4sr + \frac{8s}{p}}}{2}$$

$$x \geq \frac{1-s}{2} - \frac{1}{2}\sqrt{[1-(1+r)s]^2 + 4s\left(\frac{2}{p}-\frac{1}{16}-\frac{r}{16}\right)}$$

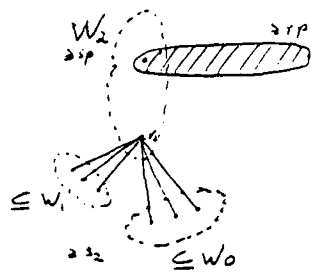
Since $rs \geq \frac{1}{K} > \frac{1}{16}$ and $r > \frac{1}{16}$ we have $\left(\frac{2}{p}-\frac{1}{16}-\frac{r}{16}\right) < 0$ for $p > 30$.

Hence $x > rs$. Let $s_1 = x$.



Every $x \in \cup W_j$ is in a maximal clique $\geq rp$ and there are $(s_1p + sp)(rp-2)$ edges to points not in $\cup W_j$

$$\frac{(s_1+s)\,(r-\frac{2}{p})}{1-s_1-s} = s_2.$$



Thus $s_2p + sp$ vertices of $W_0 \cup W_1 \cup W_2$ don't contain a triangle and there are $(s_2p+sp)(p-3)$ edges to vertices not in $\cup W_j$:

$$\frac{(s_2+s)\left(r-3/p\right)}{1-s_2-s} = s_3$$

In general

$$\frac{(s_n+s)\left(r-\frac{n+1}{p}\right)}{1-s_n-s} = s_{n+1}$$

$$s_{n+1} - s_n = \frac{(r+s)s_n + rs + s_n^2 - s_n - (s_n+s)\frac{n+1}{p}}{1-s-s_n}$$

Let $s_n = rs + d_n$ and note that $s_n > s_1 \ge rs$ and thus $d_n > 0$

Then $A = (r+s)s_n + rs + s_n^2 - s_n$

$$= (r+s+rs)rs + d_n^2 + (r+s+2rs-1)d_n$$

Now $0 < d = rs - \frac{1}{16} \le 1/4$ and thus

$\sqrt{rs} = \sqrt{\frac{1}{16}+d} \ge 1/4 + \frac{3}{2}d$

Also $r+s \ge 2\sqrt{rs} \ge \frac{1}{2} + 3d$

Thus $A \ge \left(\frac{1}{2} + \frac{1}{16} + 4d\right)\left(\frac{1}{16}+d\right) + d_n^2 + \left(\frac{1}{2}+\frac{1}{8}+5d-1\right)d_n$

$$= \frac{9}{16^2} + \frac{13}{16}d + 4d^2 + (5d - 3/8)d_n + d_n^2$$

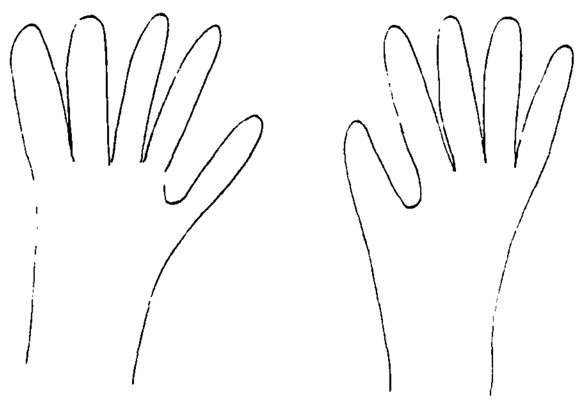
$$= \left(\frac{3}{16} - d_n\right)^2 + \frac{13}{16}d + 4d^2 + 5d\,d_n$$

$$\ge 2h \quad \text{for some } h > 0.$$

Choose $p \ge h^2$ then for $n+1 \le \frac{1}{h}$ we have

$$s_{n+1} - s_n = \frac{A-(s_n+s)\frac{n+1}{p}}{1-s-s_n} \ge \frac{2h - \frac{n+1}{p}}{1-s-s_n} \ge \frac{h}{1-s-s_n} > \frac{4}{3}h.$$

Now $s_n > s_1 + (n-1)\frac{4}{3}h$ and $s_n \ge 1-s$ for some n with $n+1 < \frac{1}{h}$, which contradicts $s_n < 1-s$.



There exists an $h > 0$ such that
$s_{n+1} - s_n \ge \frac{4}{3}h$ for $p \ge h^2$ and $n+1 \le \frac{1}{h}$.
Hence $s_k \ge 1-s$ for some k with $k+1 \le 1/h$ which contradicts that $s_n < 1-s$.

On the Product of the Independence Domination Numbers of a Graph and its Complement

Gerd H. Fricke
Wright State U.

For a graph G let

$$i(G) = \min \{ |S| : S \text{ is a maximal independent set} \}$$
$$= \min \{ |S| : S \text{ is a dominating independent set} \}.$$

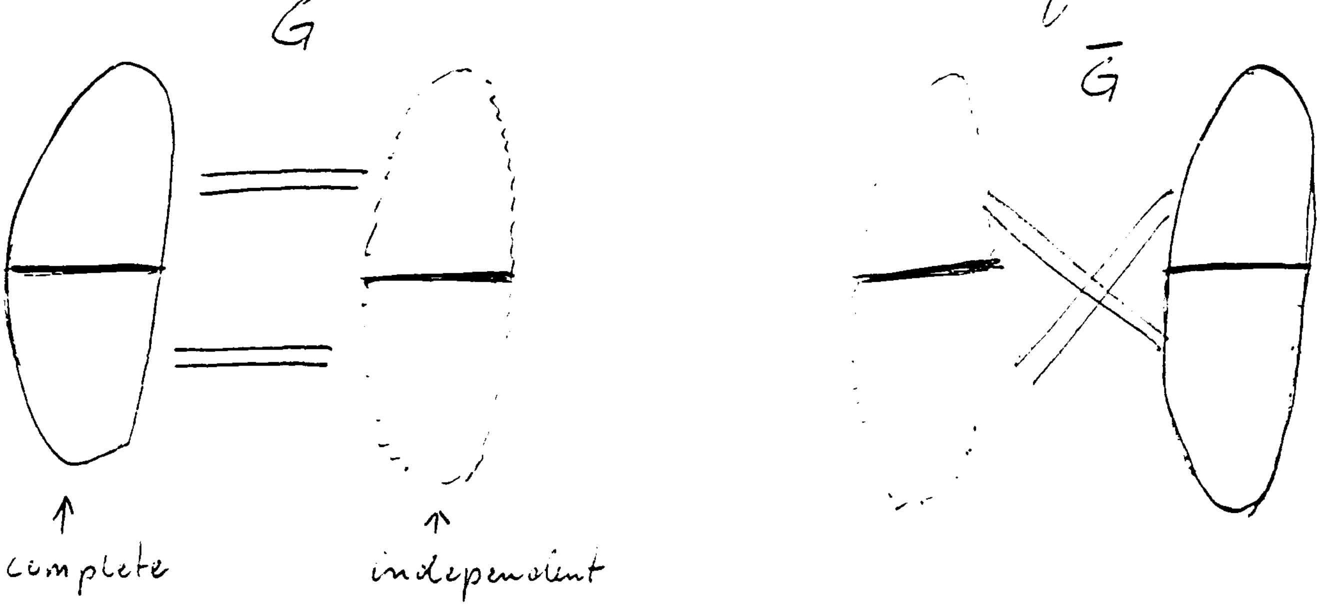
$i(\bar{G})$ is the smallest cardinality of a maximal clique in G.

Let $$mii(p) = \max_{|G| = p} \{ i(G)\, i(\bar{G}) \}.$$

It is easy to show that $mii(p)$ is nondecreasing.

Also $$\left\lfloor \frac{(p+3)^2}{16} \right\rfloor \leq mii(p)$$

Let $p = 4m$ and consider the following graph G



$i(G) = m+1$ $\qquad$ $i(\bar{G}) = m+1$

$$mii(p) \geq \frac{(p+4)^2}{16}$$

Now, $i(G) + i(\bar{G}) \leq p+1$ and thus

$$\left\lfloor \frac{(p+3)^2}{16} \right\rfloor \leq mii(p) \leq \frac{(p+1)^2}{4}$$

Recently Cockayne, Favaron, Li, and MacGillivray showed that

$$mii(p) \le \min\left\{ \frac{(p+3)^2}{8}, \frac{(p+8)^2}{10.8} \right\}.$$

Theorem: Let $0 < K < 16$ then there exists an integer p_0 such that

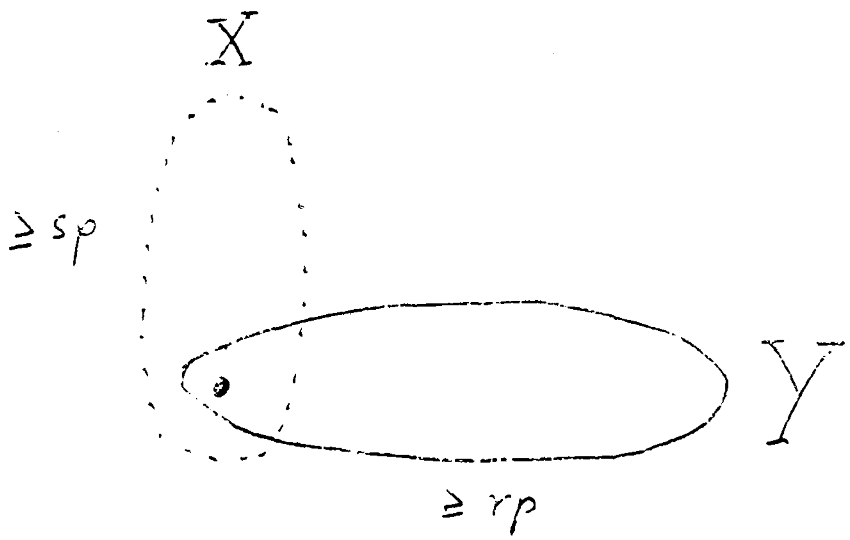
$$mii(p) \le \frac{p^2}{K} \quad \text{for all } p \ge p_0.$$

$$\left(\lim_{p \to \infty} \frac{mii(p)}{p^2} = \frac{1}{16} \right)$$

Proof: Let $0 < K < 16$ and let G be a graph of p vertices such that $i(G)\, i(\bar{G}) > \frac{p^2}{K}$.

Note that any vertex is contained in an independent set of size $\geq i(G)$ and a clique of size $i(\bar{G})$.

Let $sp = i(G)$ and $rp = i(\bar{G})$ and assume $r \leq s$.

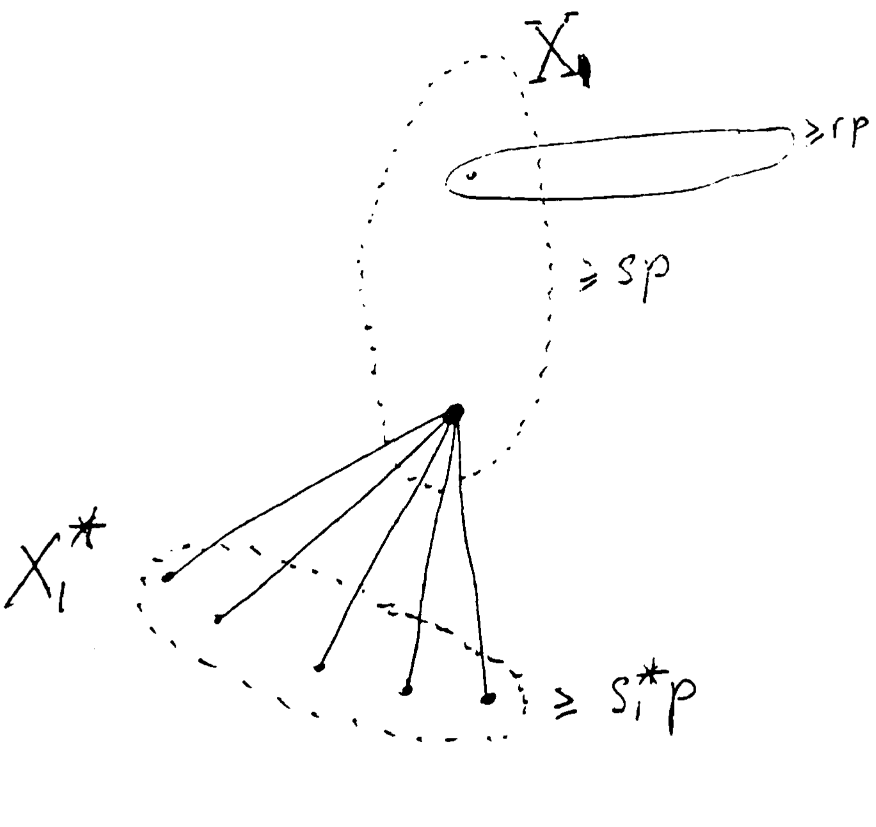


X independent

Y complete

Each vertex in X is contained in a complete K_{rp} and is adjacent to at least $rp-1$ vertices in $V \setminus X$. Thus we have at least $sp(rp-1)$ edges to $p-sp$ points in $V \setminus X$.

Let $\dfrac{sp(rp-1)}{p-sp} = s_1^* p$ then there exists a point v_1 such that $v \in V \setminus X$ and is adjacent to at least $s_1^* p$ vertices in X.



Again any vertex $v \in X_1$ is contained in a maximal clique of size $\geq rp$ and that clique contains at most one vertex of X_1 (namely v) and at most one vertex of X_1^*.

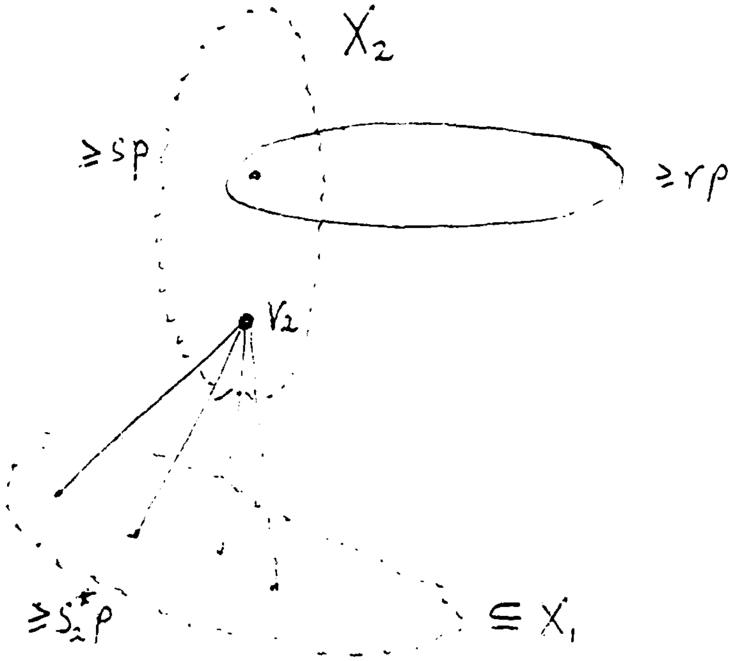
Thus every $v \in X_1$ has $rp-2$ edges to vertices not in X_1 or X_1^*.

Let $$\frac{sp\,(rp-2)}{p - s_1^* p - sp} = s_2^* p$$

Hence there exists a $v_2 \notin X_1 \cup X_1^*$ that is adjacent to $s_2^* p$ vertices of X_1

Simplify and we have

$$\frac{s\left(r - \frac{2}{p}\right)}{1 - s_1^* - s} = s_2^*$$ and repeat the argument.



Let $$\frac{s\left(r-\frac{2}{\rho}\right)}{1-s_2^*-s} = s_3^*$$ and in general

$$\frac{s\left(r-\frac{2}{\rho}\right)}{1-s_n^*-s} = s_{n+1}^*.$$

$\{s_n^*\}$ is increasing and let $\lim\limits_{n\to\infty} s_n^* = X$.

Then $$\frac{s\left(r-\frac{2}{\rho}\right)}{1-X-s} = X$$ and

$$X^2-(1-s)X+sr-\frac{2s}{\rho} = 0.$$

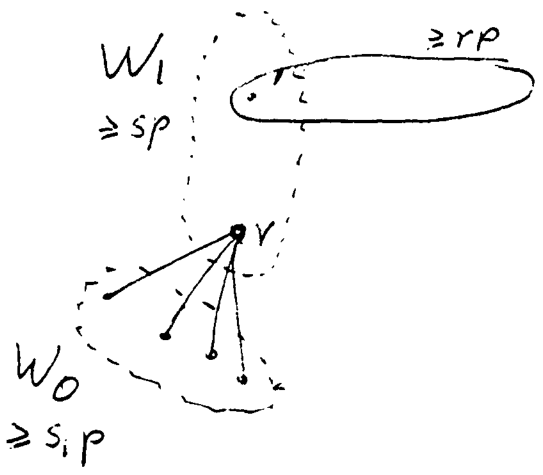
$$X = \frac{1-s-\sqrt{(1-s)^2-4sr+\frac{8s}{p}}}{2}$$

$$X \geq \frac{1-s}{2} - \frac{1}{2}\sqrt{[1-(1+r)s]^2 + 4s\left(\frac{2}{p}-\frac{1}{16}-\frac{r}{16}\right)}$$

Since $rs \geq \frac{1}{K} > \frac{1}{16}$ and $r > \frac{1}{16}$ we have

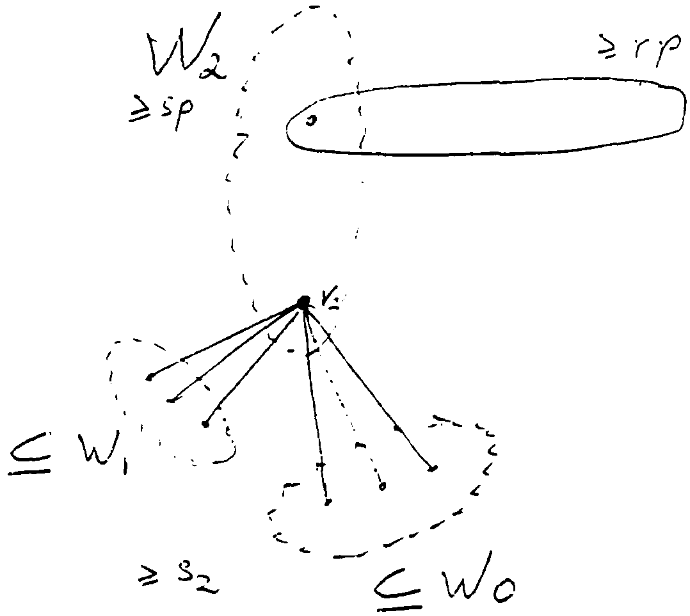
$\left(\frac{2}{p}-\frac{1}{16}-\frac{r}{16}\right) < 0$ for $p > 30$.

Hence $X > rs$. Let $s_1 = X$.



Every $x \in \cup W_j$ is in a maximal clique $\geq rp$ and there are $(s_1 p + sp)(rp-2)$ edges to points not in $\cup W_j$

$$\frac{(s_1+s)\left(r-\frac{2}{p}\right)}{1-s_1-s} = s_2.$$



Thus $s_2p + sp$ vertices of $W_0 \cup W_1 \cup W_2$ don't contain a triangle and there are $(s_2p+sp)(rp-3)$ edges to vertices not in $\cup W_j$.

$$\frac{(s_2 + s)(r - 3/p)}{1 - s_2 - s} = s_3$$

In general

$$\frac{(s_n+s)\left(r - \frac{n+1}{p}\right)}{1 - s_n - s} = s_{n+1}.$$

$$S_{n+1} - S_n = \frac{(r+s)\,S_n + rs + S_n^2 - S_n - (S_n+s)\,\frac{n+1}{p}}{1-s-S_n}$$

Let $S_n = rs + d_n$ and note that $S_2 > S_1 \geq rs$ and thus $d_2 > 0$

Then $A = (r+s)\,S_n + rs + S_n^2 - S_n$

$$= (r+s+rs)\,rs + d_n^2 + (r+s+2rs-1)\,d_n$$

Now $0 < d = rs - \frac{1}{16} \leq 1/4$ and thus

$$\sqrt{rs} = \sqrt{\tfrac{1}{16} + d} \geq 1/4 + \tfrac{3}{2}d$$

Also $r+s \geq 2\sqrt{rs} \geq \frac{1}{2} + 3d$

Thus $A \geq \left(\frac{1}{2} + \frac{1}{16} + 4d\right)\left(\frac{1}{16} + d\right) + d_n^2 + \left(\frac{1}{2} + \frac{1}{8} + 5d - 1\right)d_n$

$$= \frac{9}{16^2} + \frac{13}{16}d + 4d^2 + (5d - 3/8)\,d_n + d_n^2$$

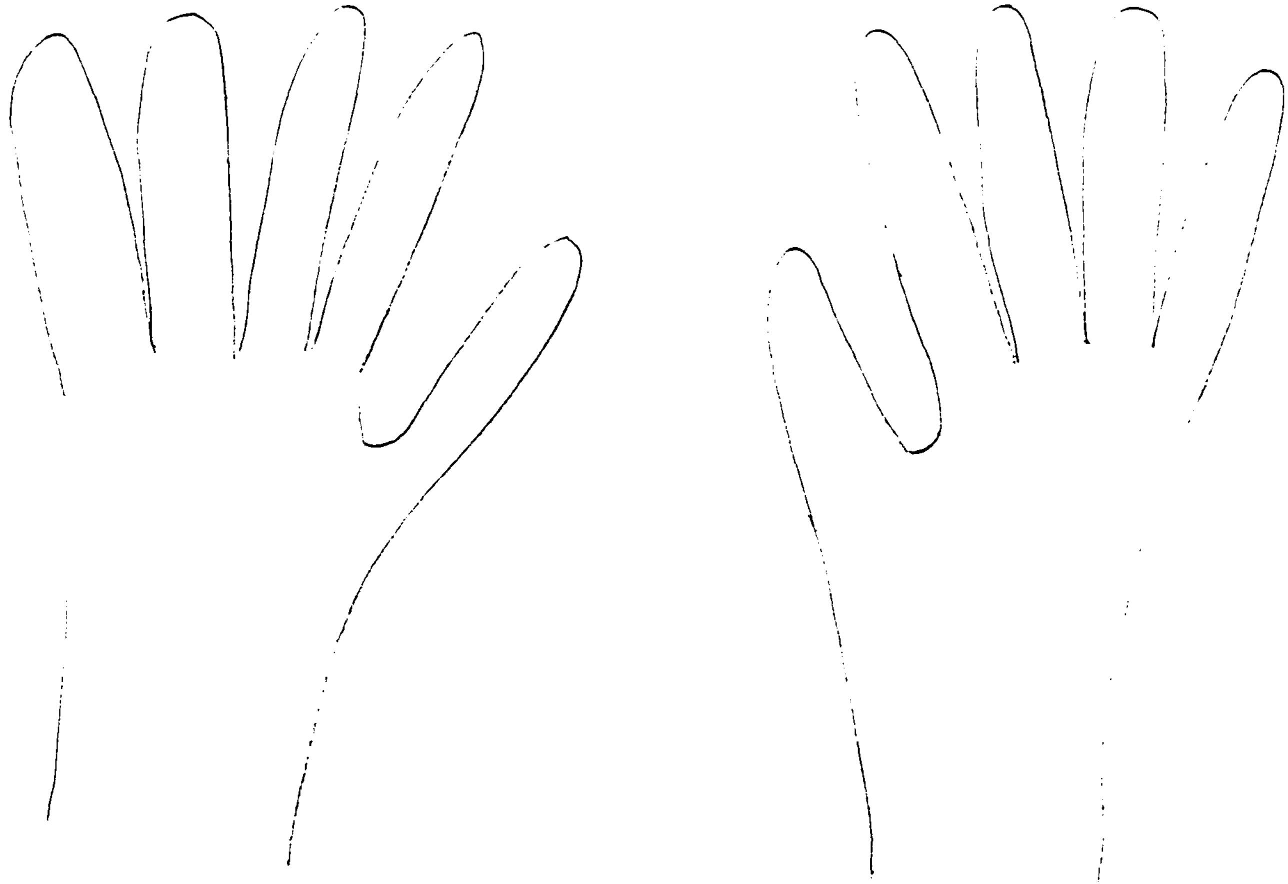
$$= \left(\frac{3}{16} - d_n\right)^2 + \frac{13}{16}d + 4d^2 + 5d\,d_n.$$

$$\geq 2h \quad \text{for some } h > 0.$$

Choose $p \geq h^2$ then for $n+1 \leq \frac{1}{h}$ we have

$$S_{n+1} - S_n = \frac{A - (S_n+s)\frac{n+1}{p}}{1-s-S_n} \geq \frac{2h - \frac{n+1}{p}}{1-s-S_n} \geq \frac{h}{1-s-S_n} > \frac{4}{3}h.$$

Now $S_n > S_1 + (n-1)\frac{4}{3}h$ and $S_n \geq 1-s$ for some n with $n+1 < \frac{1}{h}$, which contradicts $S_n < 1-s$.



There exists an $h>0$ such that

$s_{n+1} - s_n \geq \frac{4}{3} h$ for $p \geq h^2$ and $n+1 \leq \frac{1}{h}$.

Hence $s_k \geq 1-s$ for some k with $k+1 \leq 1/h$ which

contradicts that $s_n < 1-s$.

Containment of Circular-Arcs

Prof. Jeremy Spinrad
Department of Computer Science
Vanderbilt University

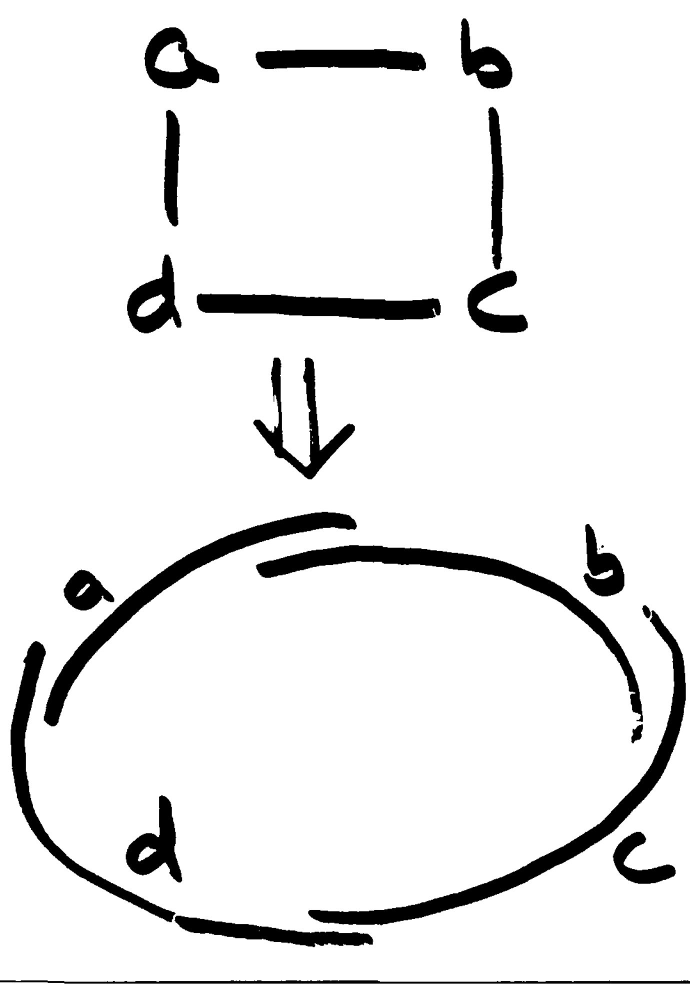
SPINRAD

Containment of Circular-Arcs

circular-arc graph

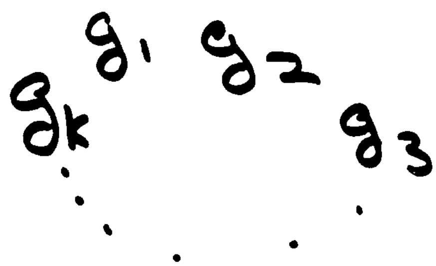
vertices ⇒ arcs on circle

x — y iff arcs intersect



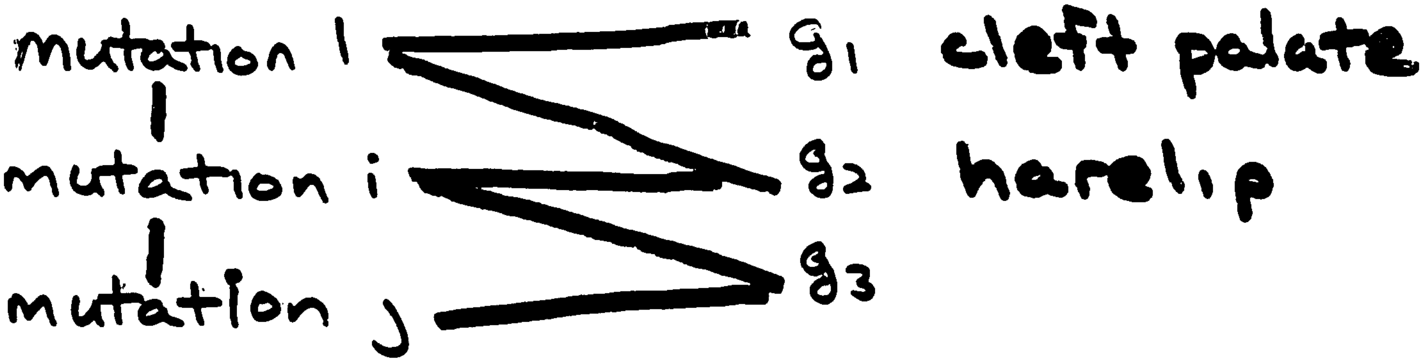
Application?

hypothesis: genes arranged in circular pattern



mutations damage consecutive portion of gene

$\Downarrow$



must be a circular-arc graph

Recognition

posed: Klee, 1969

can be used to test genetic hypothesis

solved: Tucker, 1982

difficult algorithm

also Hsu, 1990

recognition and isomorphism

Containment

$\forall x, y$ is $N(x) \subseteq N(y)$?

easy to stretch arc so

iff $N(y) \subseteq N(x)$

a/the? bottleneck step of Tucker's recognition algorithm

naive: $O(n^3)$

$O(MM)$

this talk: $O(n^2)$

General Approach

Transform to a set of bipartite problems

G circular-arc $\Rightarrow$

G' chordal bipartite

use special properties of chordal bipartite graphs to get good algorithms

Chordal Bipartite

bipartite, any cycle of length ≥ 6 has a chord

close correspondences

- β-acyclic hypergraphs
- totally balanced matrices
- strongly chordal graphs

Key Characterization [HKS]

$$\Gamma \equiv \begin{matrix} & & \vdots & & \vdots & \\ \cdots & & 1 & \cdots & 1 & \cdots \\ & & \vdots & & \vdots & \\ \cdots & & 1 & \cdots & 0 & \cdots \\ & & \vdots & & \vdots & \end{matrix}$$

G chordal bipartite $\Leftrightarrow$

$M_B(G)$ can be Γ-free ordered $\Leftrightarrow$

doubly lexical order $M_B(G)$ Γ-free

can verify that matrix is Γ-free in linear time [Lubiw]

Doubly Lexical Ordering

Input: Matrix M

$$\begin{matrix} 3 & 1 & 5 & 7 \\ 2 & 4 & 9 & 3 \\ 1 & 3 & 5 & 7 \\ 9 & 1 & 2 & 4 \end{matrix}$$

Permute so if read down/up, right/left, rows and columns ↑

$$\begin{matrix} 3 & 5 & 7 & 1 \\ 4 & 9 & 3 & 2 \\ 1 & 5 & 7 & 3 \\ 1 & 2 & 4 & 9 \end{matrix}$$

arbitrary matrices: $O(m \log n)$
[Lubiw], [Paige + Tarjan]

Graphs or 0/1 matrices: $O(n^2)$

Chordal Bipartite Containment

doubly lexical order $M_B(G)$

$$N(x) \subseteq N(y) ?$$

$$\begin{array}{ccl} & u & \\ x & 1 & \text{1st 1 in row } x \\ y & & \end{array}$$

$$N(x) \subseteq N(y) \text{ iff } y - u$$

otherwise

$$\begin{array}{ccc} & u & v \\ x & 1 & 1 \\ y & 1 & 0 \end{array} \qquad \Gamma$$

Side Issue

Nonredundant 1s

represent chordal bipartite by only those 1 values which cannot be implied by Γ-freeness

1	0	1	0
1	1	*	0
0	1	*	1

how many nonredundant 1s can there be?

Open Problems

Known $\Omega(n\log n)$ nonredundant

$O(n^{5/4+\epsilon})$

Conjecture $\Theta(n\log n)$

if true, optimal representation

How many chordal bipartite graphs?

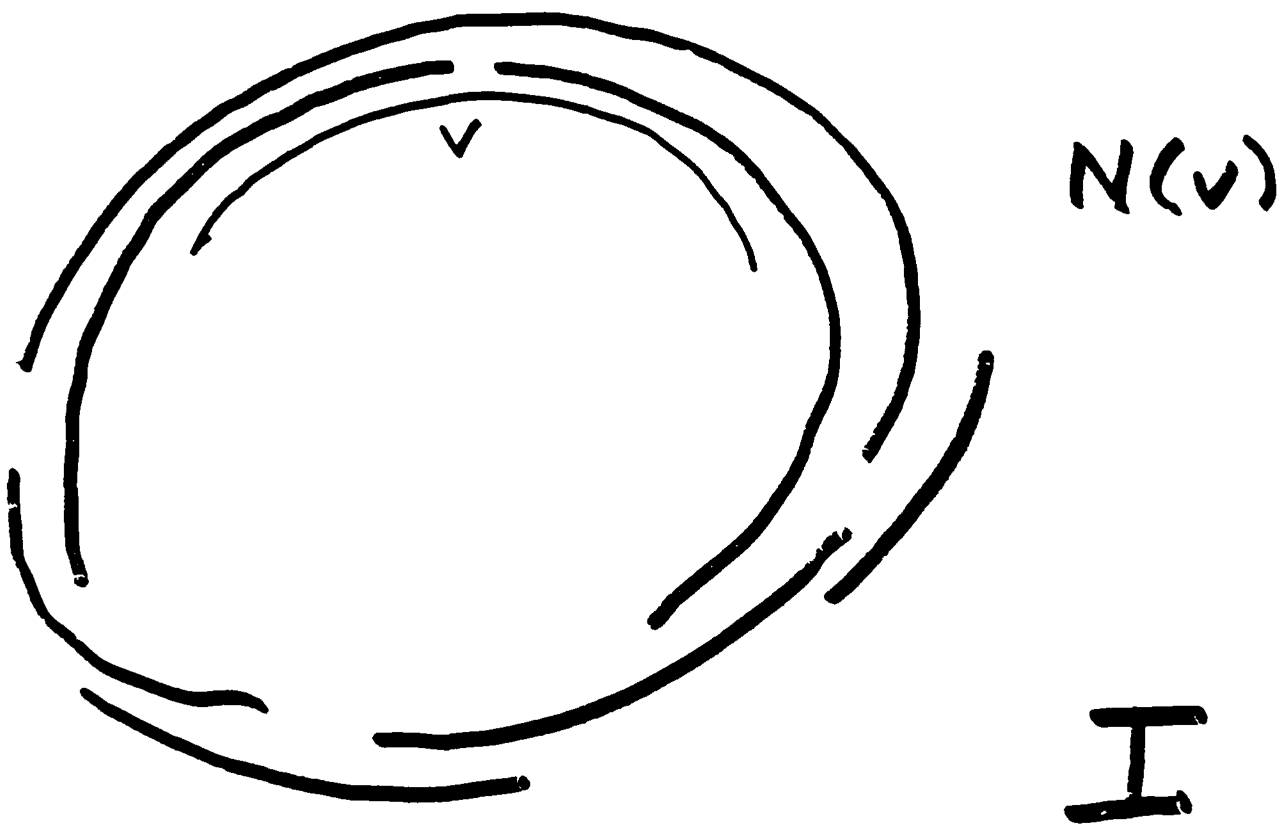
$\Omega(2^{cn\log^2 n})$

$O(2^{n^{5/4+\epsilon} c\log n})$

Spinrad's Slide

Relation to Circular-Arc Graph:

Select minimal arc v



I is an interval graph

N(v) covered by 2 cliques

Step 1

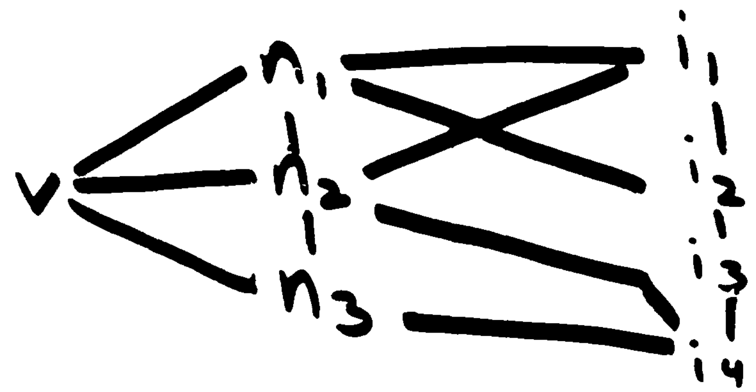
if G is circular-arc, then

$N(v)$ I

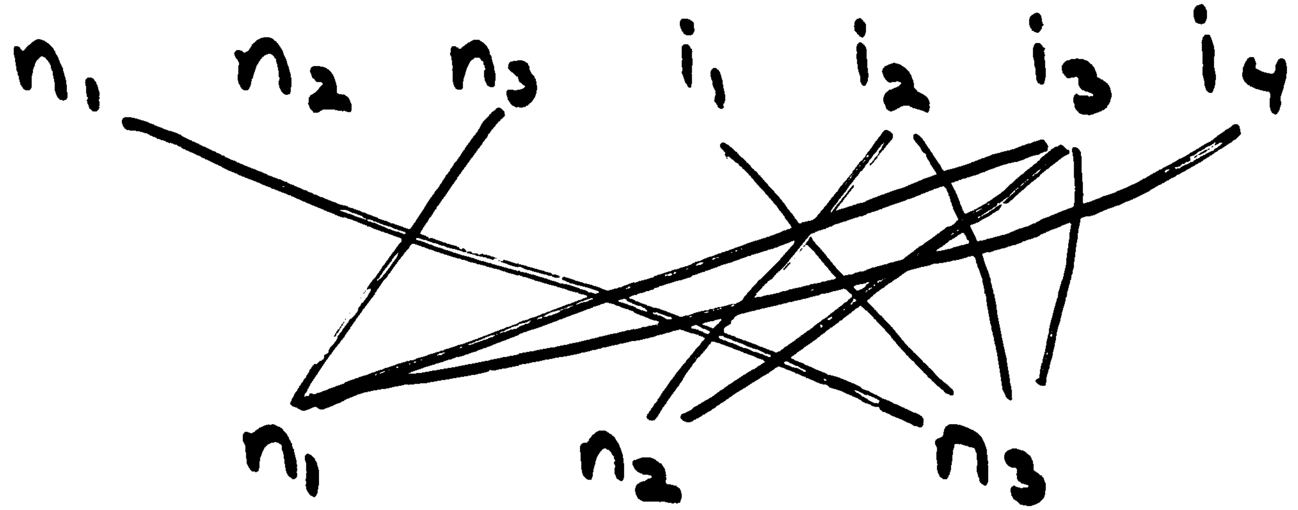
$N(v)$

bipartite complement is chordal bipartite

e.g.



⇓



"Proof"

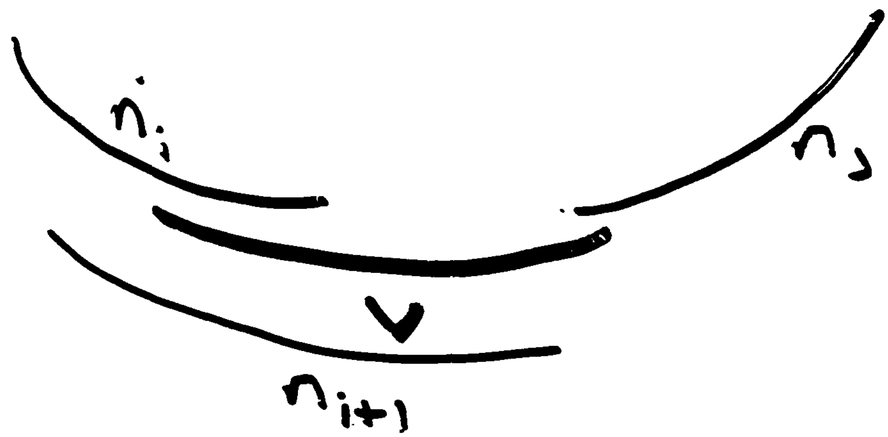
Let C be a cycle

no arcs in "bottom" N(v) contain others

top member of cycle misses exactly 2 bottom members

must be consecutive

x misses n_i, n_j



$x \in I \Rightarrow x$ misses n_{i+1}

$x \in N(v) \Rightarrow$

x misses n_{i+1}

$\Rightarrow$ could lay out cycle so neighbors of any top vertex adjacent

convex $\subseteq$ chordal bipartite

contradiction

N(v) I complement is chordal bipartite

N(v)

can compute all containment relation with respect to edges to N(v) in $O(n^3)$

Containment wrt Edges to I

1) $n_1 \in N(v) \subseteq n_2 \in N(v)$

$N(v)$ complement is

I chordal bipartite

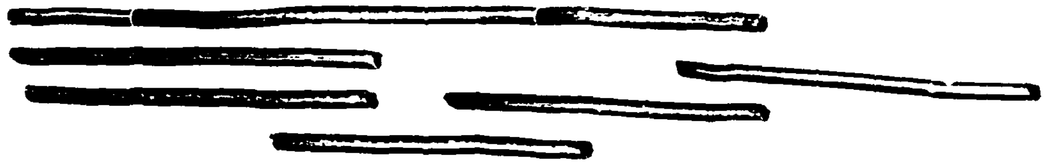
2) $i_1 \in I \subseteq i_2 \in I$

easy from "standard representation" of interval graph

3) $i_1 \in I \subseteq n_1 \in N(v)$

next slide

Lay out interval graph



"standard": start/end in same maximal clique $\Rightarrow$ same endpoint

$x \in N(v)$

'walk through' I. i startpoint,

i —— x, $i \leq x$ ~~iff~~ x's next nonneighbor ~~after~~ endpoint of

$O(nm)$?

Store only endpoints of I.

can 'walk through' I in $O(n)$ time.

$O(n^2)$

What's Next?

1) Does Tucker's algorithm run in $O(n^2)$ time?

2) Simpler $O(n^2)$ recognition

3) What else on circular arc graphs is easier than constructing representation?
   independent set

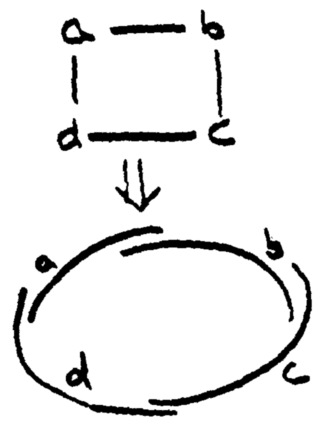
4) other uses, chordal bipartite graphs
   trapezoid graphs

Containment of Circular-Arcs

circular-arc graph

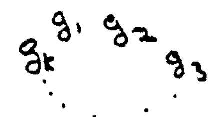
vertices ⇒ arcs on circle

$x - y$ iff arcs intersect



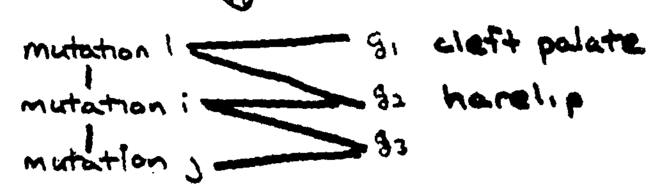
Application?

hypothesis: genes arranged in circular pattern



mutations damage consecutive portion of gene

⇓



must be a circular-arc graph

Recognition

posed: Klee, 1969

can be used to test genetic hypothesis

solved: Tucker, 1982

difficult algorithm

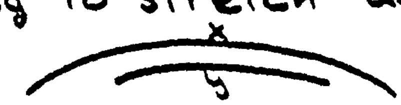
also Hsu, 1990

recognition and isomorphism

Containment

$\forall x, y$ is $N(x) \subseteq N(y)$?

easy to stretch arc so



iff $N(y) \subseteq N(x)$

a/the? bottleneck step of Tucker's recognition algorithm

naive: $O(n^3)$

$O(MM)$

this talk: $O(n^2)$

General Approach

Transform to a set of bipartite problems

G circular-arc $\Rightarrow$
G' chordal bipartite

use special properties of chordal bipartite graphs to get good algorithms

Chordal Bipartite

bipartite, any cycle of length ≥ 6 has a chord

close correspondences
- β-acyclic hypergraphs
- totally balanced matrices
- strongly chordal graphs

Key Characterization [HKS]

$$\Gamma \equiv \begin{matrix} & \vdots & & \vdots & \\ \cdots & 1 & \cdots & 1 & \cdots \\ & \vdots & & \vdots & \\ \cdots & 1 & \cdots & 0 & \cdots \\ & \vdots & & \vdots & \end{matrix}$$

G chordal bipartite $\Leftrightarrow$
$M_B(G)$ can be Γ-free ordered $\Leftrightarrow$
doubly lexical order $M_B(G)$ Γ-free

can verify that matrix is Γ-free in linear time [Lubiw]

Doubly Lexical Ordering

Input: Matrix M

$$\begin{matrix} 3 & 1 & 5 & 7 \\ 2 & 4 & 9 & 3 \\ 1 & 3 & 5 & 7 \\ 9 & 1 & 2 & 4 \end{matrix}$$

Permute so if read down/up, right/left, rows and columns $\uparrow$

$$\begin{matrix} 3 & 5 & 7 & 1 \\ 4 & 9 & 3 & 2 \\ 1 & 5 & 7 & 3 \\ 1 & 2 & 4 & 9 \end{matrix}$$

arbitrary matrices: $O(m \log n)$
[Lubiw], [Paige + Tarjan]

Graphs or 0/1 matrices: $O(n^2)$

Chordal Bipartite Containment

doubly lexical order $M_B(G)$

$N(x) \subseteq N(y)$?

$$\begin{array}{lcl} & u & \\ x & 1 & \text{1st 1 in row } x \\ y & & \end{array}$$

$N(x) \subseteq N(y)$ iff y—u

otherwise

$$\begin{array}{lcc} & u & v \\ x & 1 & 1 \\ y & 1 & 0 \end{array} \qquad \Gamma$$

Side Issue

Nonredundant 1s

represent chordal bipartite by only those 1 values which cannot be implied by Γ-freeness

$$\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 1 & 1 & * & 0 \\ 0 & 1 & * & 1 \end{array}$$

how many nonredundant 1s can there be?

Open Problems

Known $\Omega(n \log n)$ nonredundant

$O(n^{5/4+\epsilon})$

Conjecture $\Theta(n \log n)$

if true, optimal representation

How many chordal bipartite graphs?

$\Omega(2^{cn \log^2 n})$

$O(2^{n^{5/4+\epsilon} c \log n})$

Lower Bound

1 1 1 1 (diagonal)	any Γ-free
any Γ-free	*

$\text{nonredundant}(n) \geq 2nr(\frac{n}{2}) + \frac{n}{2}$

$\Omega(cn \log n)$

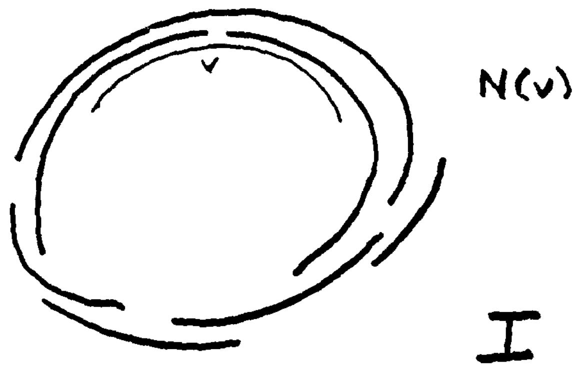
any perfect matching in upper left

$\Omega(2^{cn \log^2 n})$ graphs

if $\Theta(2^{cn \log^2 n})$, optimal

Relation to Circular-Arc Graph.

Select minimal arc v



I is an interval graph

N(v) covered by 2 cliques

Step 1

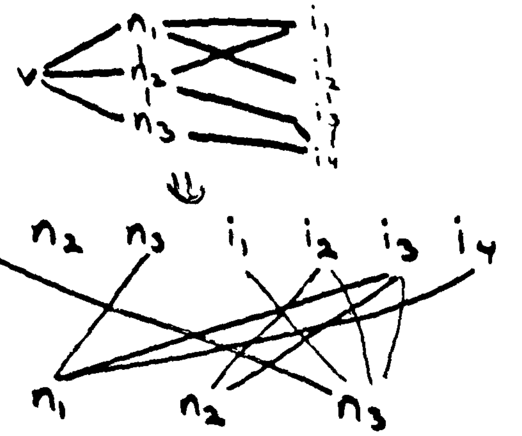
if G is circular-arc, then

N(v) I

N(v)

bipartite complement is chordal bipartite

e.g.



"Proof"

Let C be a cycle

no arcs in "bottom" N(v) contain others

top member of cycle misses exactly 2 bottom members must be consecutive

x misses n_i, n_j

$x \in I \Rightarrow x$ misses n_{i+1}

$x \in N(v) \Rightarrow$

x misses n_{i+1}

$\Rightarrow$ could lay out cycle so neighbors of any top vertex adjacent

convex $\subseteq$ chordal bipartite

contradiction

$N(v)$	I	complement is
$N(v)$		chordal bipartite

can compute all containment relation with respect to edges to $N(v)$ in $O(n^2)$

Containment wrt Edges to I

1) $n_1 \in N(v) \subseteq n_2 \in N(v)$

$N(v)$	complement is
I	chordal bipartite

2) $i_1 \in I \subseteq i_2 \in I$

easy from "standard representation" of interval graph

3) $i_1 \in I \subseteq n_1 \in N(v)$

next slide

What's Next?

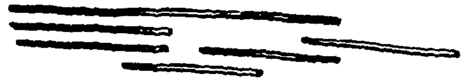
1) Does Tucker's algorithm run in $O(n^2)$ time?

2) Simpler $O(n^2)$ recognition

3) What else on circular arc graphs is easier than constructing representation?
   independent set

4) other uses, chordal bipartite graphs
   trapezoid graphs

Lay out interval graph



"standard": start/end in same maximal clique $\Rightarrow$ same endpoint

$x \in N(v)$

'walk through' I. i startpoint, i——x, $i \leq x$ iff x's next nonneighbor after endpoint of

$O(nm)$?

store only endpoints of I.

can 'walk through' I in $O(n)$ time.

$O(n^2)$

A Fast Parallel Recognition Algorithm

for

a Class of Tree–representable Graphs

Stephan Olariu

Department of Computer Science

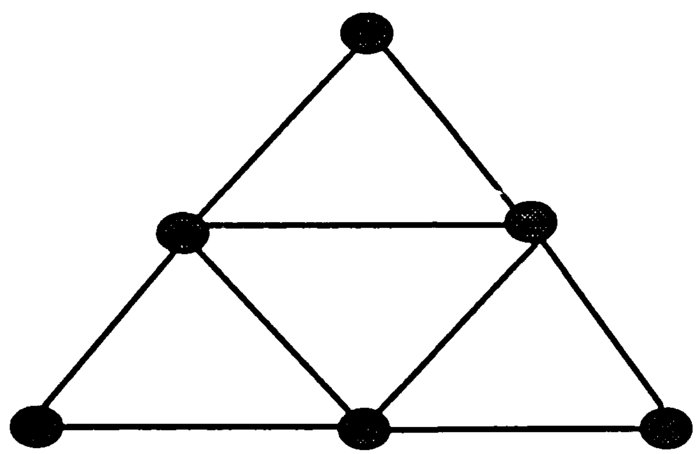
Old Dominion University

Common metrics for "local density"

- Complete graph (*clique*)
- Cliques with a "few" edges missing
- No "long" paths allowed
- A "few" long paths allowed

"long" path

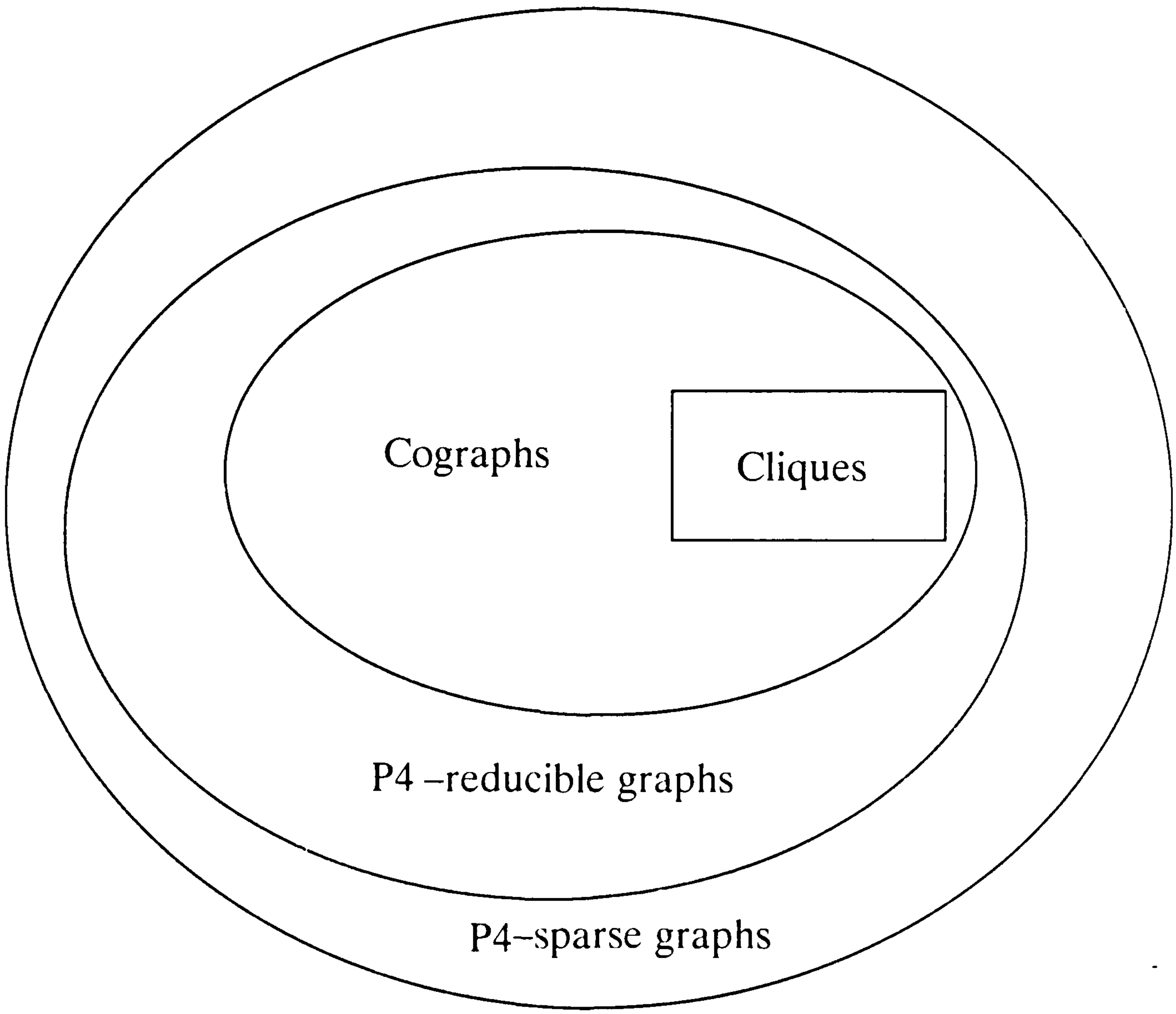
Definition A graph G is P4-sparse if no set of five vertices of G induces more than one P4.



Cographs: a class of graphs containing no P4s

P4-reducible graphs: a class of graphs such that every vertex belongs to at most one P4

Applications: scheduling, computational semantics, pattern recognition etc.



Definition **For every graph G consider the graph C(G) returned by the following procedure:**

```
Procedure Greedy(G);
{Input: an arbitray graph G;
 Output: a graph C(G)}
begin
    C(G) = G;
    while there exists a P4 in C(G) do
        pick an arbitrary P4 uvxy;
        pick z at randon in {u,y};
        C(G)=C(G)-{z};
    return(C(G))
end; {Greedy}
```

Theorem **For a graph G with no induced C5 the following statements are equivalent:**
(i) G is P4-sparse;
(ii) for every induced subgraph H of G, C(H) is unique up to isomorphism

Consider $G_1=(V_1,\varnothing)$ and $G_2=(V_2,E_2)$ $(V_1 \cap V_2=\varnothing)$ with $V_2 = \{v\} \cup K \cup R$ such that

- $|K|=|V_1|+1 \geq 2$
- K is a clique.
- Every vertex in R is adjacent to all the vertices in K and non-adjacent to v.
- There exists a vertex v' in K such that $N_{G_2}(v)=\{v'\}$ or $N_{G_2}(v)=K-\{v'\}$.

Choose a bijection $f\colon V_1 \to K-\{v'\}$ and define

$$G_1 \textcircled{2} G_2=(V_1 \cup V_2, E_2 \cup E')$$

with

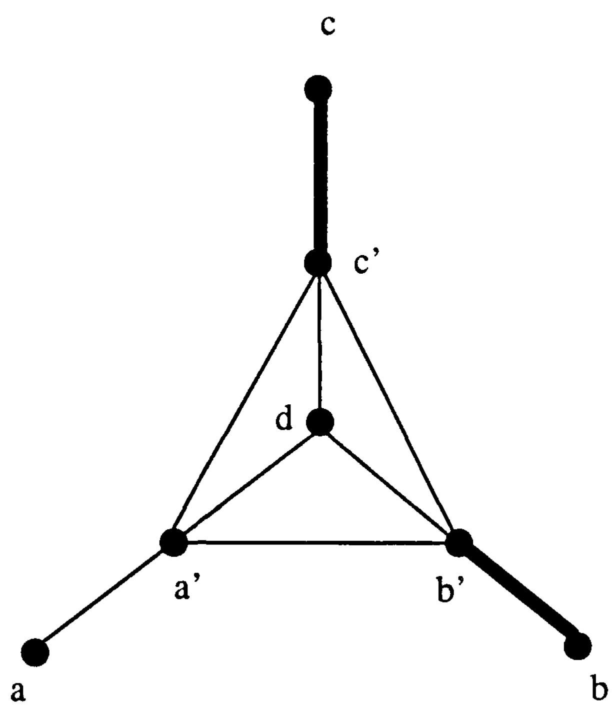
$$E' = \begin{cases} \{xf(x) \mid x \in V_1\} & \text{whenever } N_{G_2}(v)=\{v'\} \\ \\ \{xz \mid x \in V_1,\ z \in K-\{f(x)\}\} & \text{whenever } N_{G_2}(v)=K-\{v'\} \end{cases}$$

Theorem *G is a P4-sparse graph if, and only if, G is obtained from single-vertex graphs by a finite sequence of operations* $\textcircled{0}$, $\textcircled{1}$, $\textcircled{2}$. □

```
Procedure Build_tree(G);
{Input: a P4-sparse graph G=(V,E);
 Output: the ps-tree T(G) corresponding to G.}
 begin
   if |V| = 1 then
       return the tree T(G) consisting of the unique vertex of G;
   if G (Ḡ) is disconnected then begin
        let G1, G2,..., Gp (p≥2) be the components of G (Ḡ);
        let T1, T2, ...,Tp be the corresponding ps-trees rooted at r1, r2, ..., rp;
        return the tree T(G) obtained by adding r1, r2,..., rp as children of a node
        labelled 0 (1);
        end
    else begin {now both G and Ḡ are connected}
        write G = G1 ② G2
        let T1, T2 be the corresponding ps-trees rooted at r1 and r2;
        return the tree T(G) obtained by adding r1, r2 as children of a node labelled 2
        end
 end; {Build_tree}
```

An example...

G



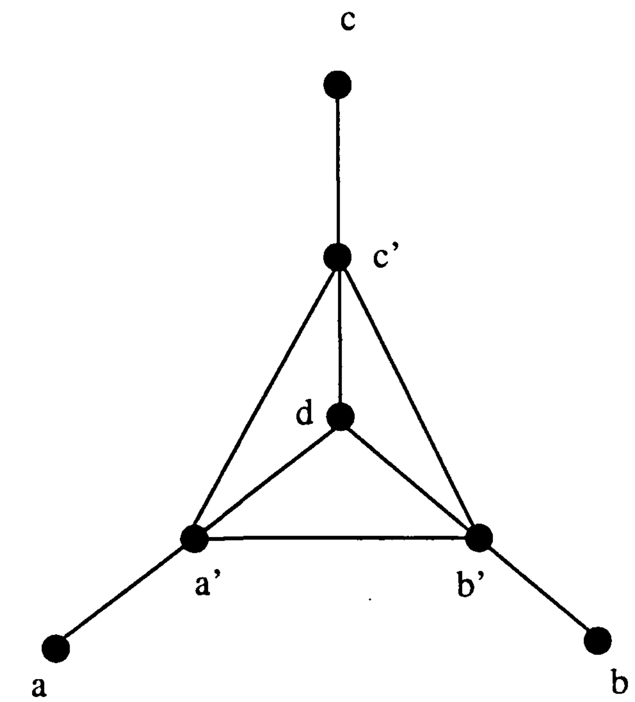
G_1 $(\{b,c\},\Phi)$

G_2 $(\underbrace{\{a\}}_{\mathbf{v}}\cup\underbrace{\{a',b',c'\}}_{\mathbf{v'}}\cup\{d\},\{aa',a'b',a'c',b'c',a'd,b'd,c'd\})$

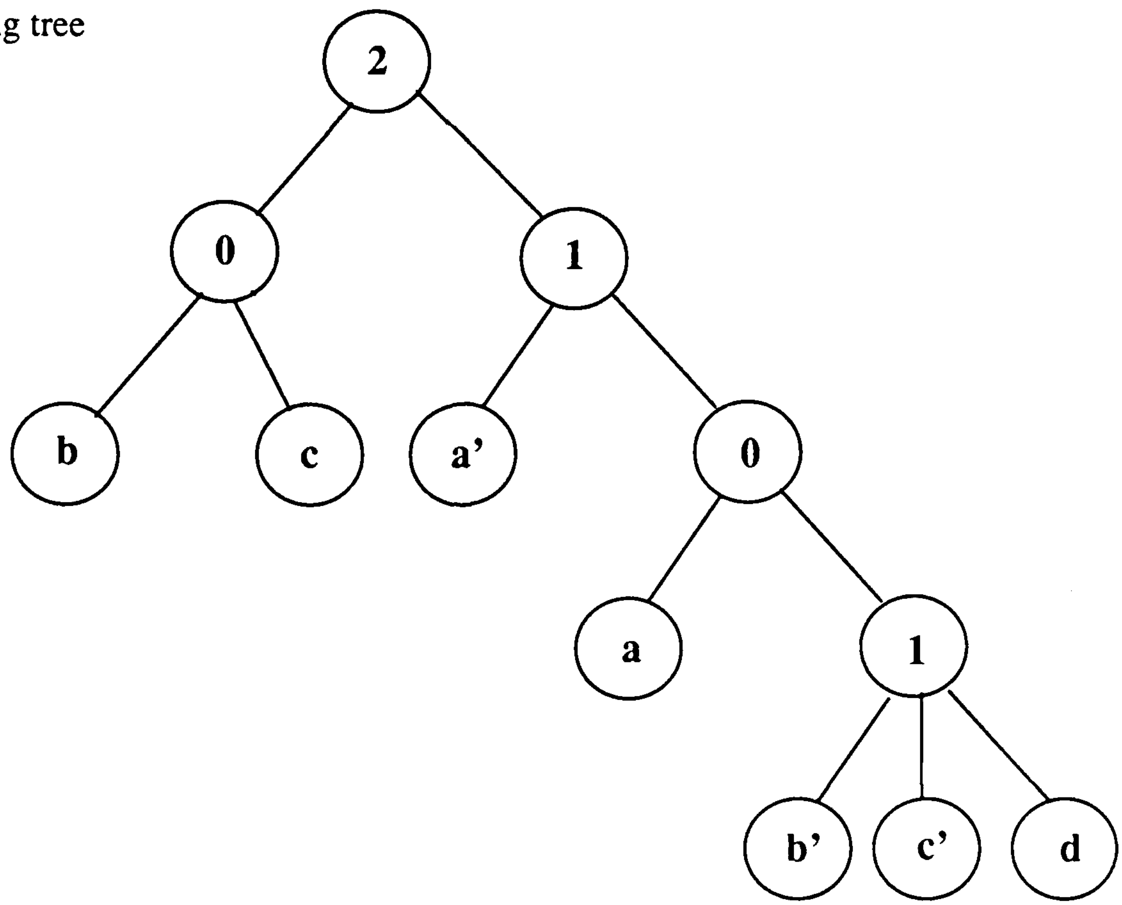
K **R**

An example...

a P4–sparse graph



the corresponding tree



A set C of vertices of G is termed *regular* if it admits a partition into non-empty, disjoint sets K and S satisfying the following conditions:

(r1) $|K|=|S|\geq 2$, S stable, K a clique;

(r2) every vertex in V−C belongs to precisely one of the sets:

$T(C)=\{x\in V-C \mid x \text{ adjacent to all the vertices in } C\}$;

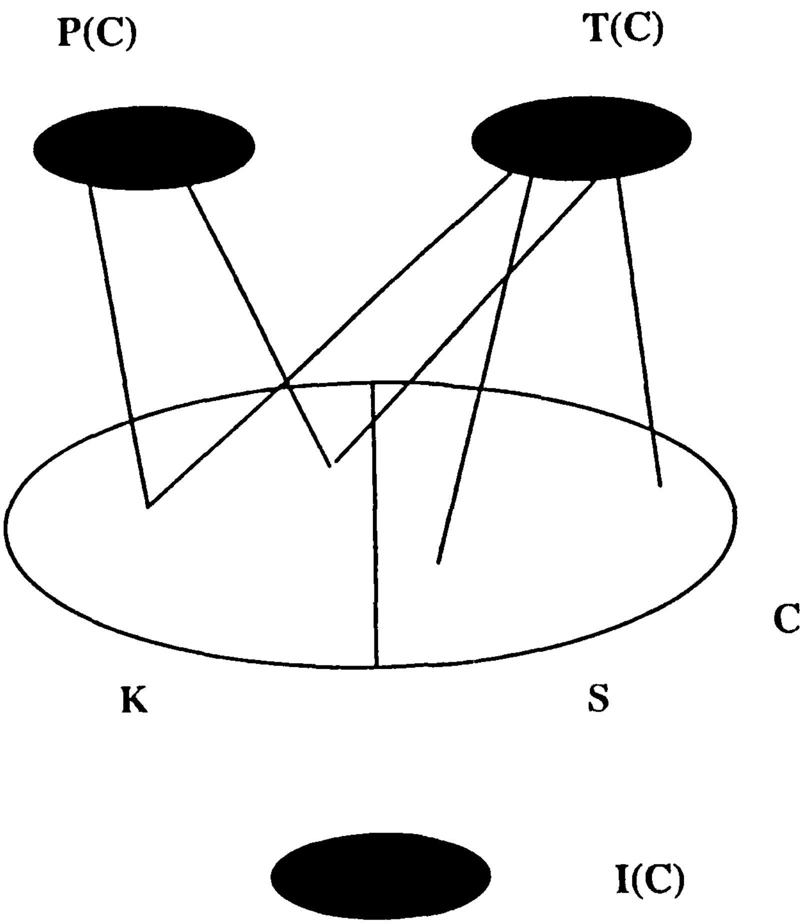
$I(C)=\{x\in V-C \mid x \text{ non-adjacent to all the vertices in } C\}$;

$P(C)=\{x\in V-C \mid x \text{ adjacent to all the vertices in } K \text{ and non-adjacent to all the vertices in } S\}$.

(r3) there exists a bijection $f:S\rightarrow K$ such that

either $N(x)\cap K=\{f(x)\}$ for every x in S,

or else $N(x)\cap K=K-\{f(x)\}$ for every x in S.



Let G = (V,E) be an *arbitrary* graph.

Fact (*Regularity is hereditary*)

Let C = (K,S,f) be a regular set in G and let Z be a subset of S with $|Z| < |S| - 2$. Then $C' = C - \{x, f(x) \mid x \in Z\}$ is regular

Fact (*Containment)*

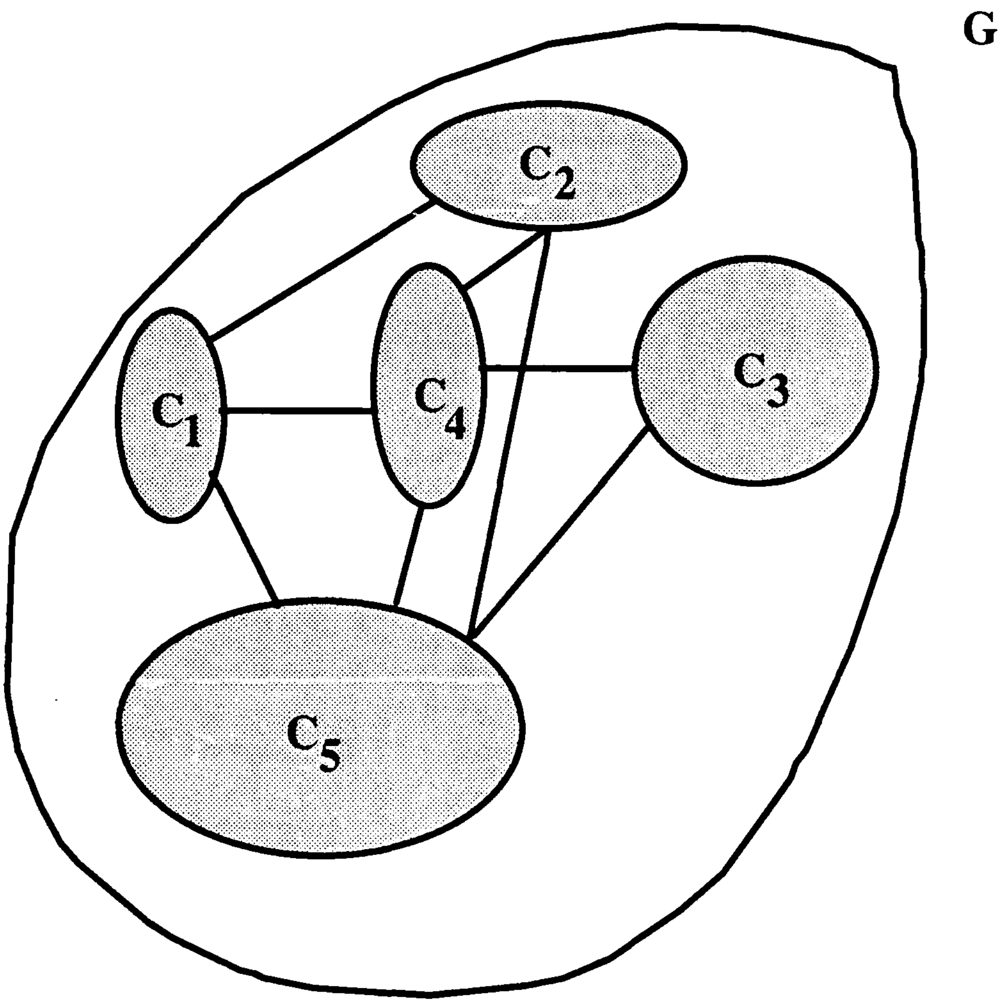
Let C = (K,S,f) be regular. For every pair of distinct u, v in C with $u \neq f(v)$ and $v \neq f(u)$, the unique P4 containing u and v belongs to C

Fact (*Black hole property)*

A regular set is maximal if, and only if, every regular P4 containing a vertex in C is included in C

Fact (*Separation property)*

Two maximal regular sets coincide whenever they intersect



The "world " of regular sets

Given an arbitrary graph G construct a graph G^* as follows:

remove in every maximal regular set $C = (K,S,f)$ all the vertices in S except for an arbitrary one

Theorem For every graph G, the graph G^* is unique up to isomorphism

Theorem For an arbitrary graph G the following statements are equivalent:

(i) G is P4-sparse;

(ii) G^* is a cograph

Algorithm Recognize(G);
{Input: an arbitrary graph G;
Output: "yes" or "no" depending on whether or not
G is P4–sparse}

Step 1. Find all maximal regular sets in G;

Step 2. Compute G^*;

Step 3. **if** G is a cograph **then**
return("yes")
else
return("no")

Step 4. Stop.

Our algorithm:

O(log n) EREW time using $O\left(\frac{n^2 + mn}{\log n}\right)$ **processors**

What we do:

- **recognize P4-sparse graphs;**
- **construct the corresponding tree**

The Algorithm

- The EREW model of computation is assumed;
- G is an arbitrary graph represented by adjacency lists;
- for every vertex x, assign one processor to every entry on the adjacency list of x;
- the vertices are enumerated as $v_1, v_2, \ldots, v_n$ in a way that will be explained later;
- sets will be represented by their characteristic vector;
  - * computing the cardinality of a set takes O(log n) time using O(n/log n) processors;
  - * given sets S, S' of vertices of G, computing S - S', $S \cup S'$, $S \cap S'$, as well as testing $S = \emptyset$, $S \subseteq S'$ takes O(log n) time using O(n/log n) processors;
- to compute N[x] we need O(log n) time and O(n/log n) processors.

Processor assignment:

- for every x of G, every entry on the adjacency list of x receives one processor;

- every edge e_i $(i=1,2,\ldots,m)$ receives $1+\lceil n/\log n \rceil$ processors

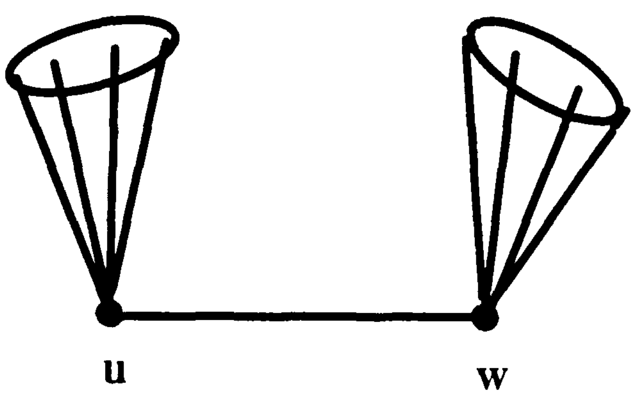
$P(e_i, 0), P(e_i, 1), \ldots P(e_i, m)$

Note: the total number of processors is bounded by

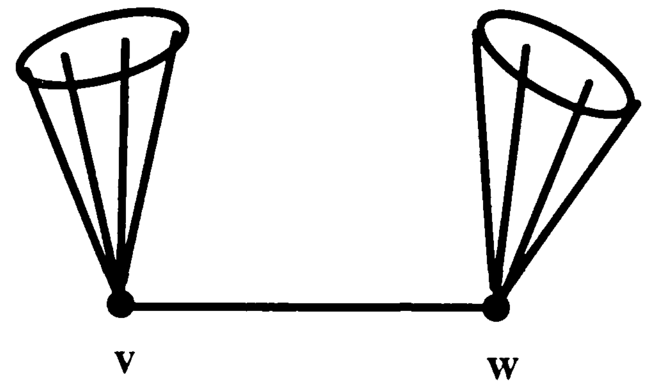
$$O\left(\frac{mn}{\log n}\right)$$

How do we find a regular set?

$N_{uw} = N[u]-N[w]$ $\qquad$ $N_{wu} = N[w]-N[u]$



$N_{vw} = N[v] - N[w]$ $N_{wv} = N[w] - N[v]$



Fact The edge vw is the *midedge* of a regular P4 in G only if $|N_{vw}| = |N_{wv}| = 1$, and $uz \notin E$ with u, z standing for the unique vertex in N_{vw} and N_{wv}, respectively

- For every edge e =vw the sets N_{vw} and N_{wv} can be computed in O(log n) time as follows:

  - N[v] will be broadcast to all $d_G(v)$ edges incident with v.

  - N[w] will be broadcast to all $d_G(w)$ edges incident with w.

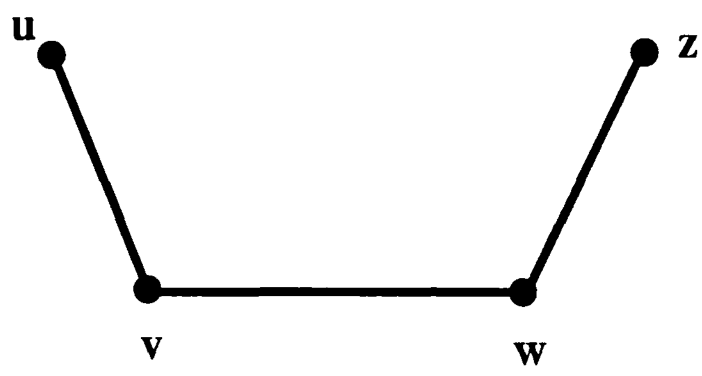
Note: Total number of processors $O(\frac{n}{\log n} \sum d_G(v)) = O(\frac{mn}{\log n})$

```
Procedure Find_Regular_P4s(G);
begin
   for every edge e_i={v,w} of G do in parallel begin
      N_vw ← N[v]−N[w];
      N_wv ← N[w]−N[v];
      if |N_vw|=|N_wv|=1 then {let N_vw={u}, N_wv={z}, U={u,v,w,z}}
         if uz∉E then begin
            for all the vertices x in V−{u,v,w,z} do in parallel
               if x∉T(U)∪P(U)∪I(U) then
                  some processor P(e_i,t(≠0)) writes a "1" in its own memory;
            if no "1" was written then P(e_i,0) does the following
                  - remembers {u,v,w,z};
                  - flags itself
         end {if}
   end {for}
end; {Find_Regular_P4s}
```

Fact *Procedure Find_Regular_P4s correctly computes the set of all the regular P_4s in G in O(log n) EREW time using* $O(\frac{n^2+mn}{\log n})$ *processors.*

More terminology...

regular P4



Assume $u = v_j$ and $z = v_k$ with $j < k$

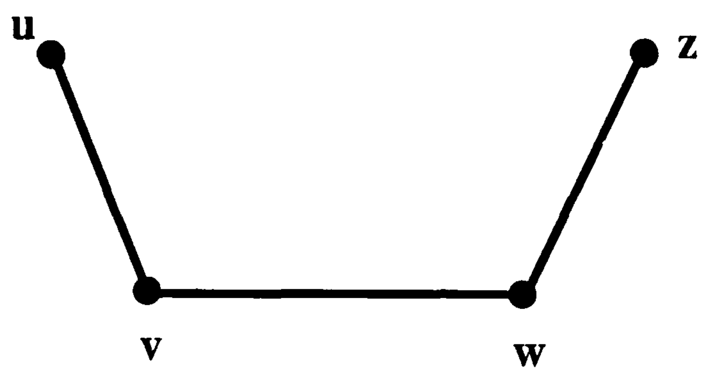
u is local "loser"

v is local "winner"

- Each flagged processor P(i) writes the identity of the local loser and winner into A[i] and B[i], respectively (A and B are one–dimensional arrays of m elements initialized to 0)
- Sort all non–zero entries of A and B and remove duplicates: this takes O(log n) time using O(m) procesors
- Construct a bitvector L: bit i of L is set to 1 iff v_i is a local loser (This takes O(log n) time and O(n/log n) processors)

More terminology...

regular P4



- An endpoint u of a regular P4 is called a *global winner* if the bit corresponding to u in L is 0

- To record all the global winners we construct a bitvector W using the information in the array B;
  (This takes O(log n) time and O(n/log n) processors)

- W = W – L (this is the set of all global winners)

- Every flagged processor corresponding to a global winner is referred to as *essential*

```
Procedure Find_Winners_and_Losers(G);
begin
    A[1:m]←B[1:m]←0;
    L←W←0;
    for every flagged processor P(i) in parallel begin
        A[i]← local loser corresponding to e_i;
        B[i]← local winner corresponding to e_i
        end; {for}
    let A[1], A[2],...A[k] be the non-zero entries of A
    in sorted order with all duplicates removed;
    let B[1], B[2],...B[l] be the non-zero entries of B
    in sorted order with all duplicates removed;
    for all i←1 to k do in parallel
        set the A[i]-th bit of L to 1;
    for all i←1 to l do in parallel
        set the B[i]-th bit of W to 1.
    W←W−L; {find global winners}
    broadcast W to all the processors P(i);
    for every flagged processor P(i) in parallel
        if the local winner of e_i is in W then
            P(i) does the following:
                - remembers that its local winner is a global winner;
                - marks itself as "essential"
    return(L,W)
end; {Find_Winners_and_Losers}
```

Fact *Procedure Find_Winners_and_Losers correctly computes the set of all the global winners and losers in* $O(\log n)$ *EREW time using* $O(\frac{mn}{\log n})$ *processors.*

```
Procedure Construct_SK(G);
begin
    let w1, w2, ..., wp stand for the global winners;
    for i←1 to p do in parallel
        if processor P(i) is essential then begin
            processor P(i) sets to 1 the bit of Si cooresponding to wi;
            let P(i1), P(i2), ..., P(iti) (1≤i≤p) be the
            essential processors whose local winner is wi;
            for j←1 to ti do in parallel
                processor P(ij) sets the k-th bit of Si with
                vk standing for its local loser;
            processor P(i1) broadcasts to P(i2), ..., P(iti)
            the identity of the two midpoints it stores;
            for j←2 to ti do in parallel
                processor P(ij) marks the midpoint it stores coinciding
                with one of the midpoints received;
            for j←1 to ti do in parallel
                processor P(ij) sets to 1 the bit of Ki corresponding
                to its unmarked midpoint;
            ri←|Ki|-|Si|
            if |N(wi)∩Ki|=1 then
                fi(wi)← the unique vertex in N(wi)∩Ki
            else
                fi(wi)← the unique vertex in Ki-N(wi)
        end; {if}
    return(SK(G))
end; {Construct_SK}
```

To summarize our previous discussion, we state the following result.

Fact *Procedure Construct_SK correctly computes the information in every SK[i] ($1 \le i \le p$) in $O(\log n)$ time using $O(\frac{n^2}{\log n})$ processors in the EREW-PRAM model.* □

Procedure Recognize_P4sparse(G);
{Input: an arbitrary graph G,E) with |V|=n and |E|=m;
Output: "yes" or "no" depending on whether or not G is a P_4–sparse graph;}

```
0. begin
1.     Find_Regular_P4s(G);
2.     Find_Winners_and_Losers(G);
3.     using the information contained in L construct the graph G*;
4.         if Cograph(G*) then
5.             return("yes");
6.     return("no")
7. end; {Recognize_P4sparse}
```

Theorem *Procedure Recognize_P4sparse correctly determines whether an arbitrary graph G=(V,E) with |V|=n and |E|=m is a* P_4–sparse *graph in O(*log n*) time using* $O(\frac{n^2+mn}{\log n})$ *processors in the EREW-PRAM model.*

Constructing the tree representation of P4–sparse graphs

- $T(G)$, the cotree of the reduced graph G^* is available as a byproduct of Cograph(G^*)

- for convenience we enumerate the maximal regular sets as

$$C_1 = (K_1, S_1, f_1),\ C_2 = (K_2, S_2, f_2),\ \ldots,\ C_p = (K_p, S_p, f_p),$$

- at the end of the successful recognition of a P4–sparse graph G, the relevant information about G is stored in the tuple $(T(G), SK(G))$

What is SK(G)??

- We can think of SK(G) as a 1–dimensional array such that S[i] contains the following information

  - characteristic vectors of K_i and S_i

  - the identity of the unique vertex w_i in S_i that belongs to G*

  - the identity of $f_i(w_i)$

  - $r_i = |K_i| = |S_i|$

  Let w , w , ..., w be the global winners as recorded in W

- To compute S_i every essential processor whose local winner is w_i sets the j–th bit of S_i, with j standing for its local loser

- To compute K_i we do the following

  - In O(log n) time identify the subset $P(i_1)$, $P(i_2)$, ..., $P(i_{t_i})$ of essential processors whose local winner is $v_{W[i]}$

  - Processor $P(i_1)$ broadcasts to $P(i_2)$,..., $P(i_{t_i})$ the identity of the midpoint it has remembered

  - Every processor $P(i_j)$ marks its own midpoint coinciding with the one received by broad-casting

  - Every processor $P(i_j)$ sets to 1 the bit of K_i corresponding to the unmarked midpoint it stores

```
Procedure Parallel_Build_ps_Tree(G);
{Input: a P4-sparse graph represented as (T(G),SK(G))
 Output: the corresponding ps-tree T(G), rooted at R;}
begin
    for every essential processor P(i) do in parallel begin
        create a 2-node β;
        create a 1-node γ;
        add γ as a child of β;
        add λ as a child of γ;
        if r_i=2 then begin
            add the unique vertex in S_i−{w_i} as a child of β;
            add f_i(w_i) as a child of γ
            end
        else begin
            create a 0-node α;
            add α as a child of β;
            add all vertices in S_i−{w_i} as children of α;
            if w_i is adjacent to f_i(w_i) then
                add f_i(w_i) as a child of γ
            else
                add all vertices in K_i−f({w_i}) as children of γ
            end; {if}
        if d(λ')≠|N(w_i)∩K_i|+1 then
            add β as a child of λ'
        else begin
            add β as a child of p(λ');
            delete λ'
            end {if}
    end; {for}
    if d(R)=1 then R←unique child of R;
    return(T(G))
end; {Build_ps_Tree}
```

Theorem *Procedure Parallel_Build_ps_Tree correctly constructs the ps-tree of a* P_4–sparse *graph* $G=(V,E)$ *with* $|V|=n$ *and* $|E|=m$ *in* $O(\log n)$ *EREW time using* $O(\frac{n}{\log n})$ *processors.*

A Fast Parallel Recognition Algorithm

for

a Class of Tree–representable Graphs

Stephan Olariu

Department of Computer Science

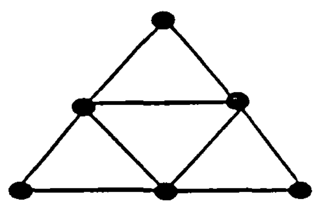
Old Dominion University

Common metrics for "local density"

- Complete graph (*clique*)
- Cliques with a "few" edges missing
- No "long" paths allowed
- A "few" long paths allowed

"long" path

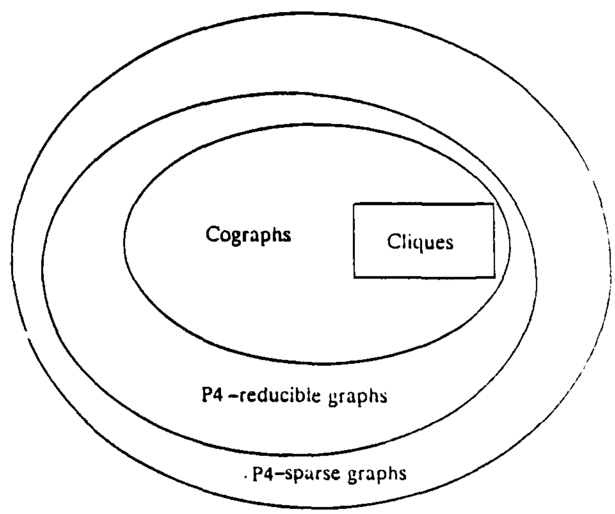
Definition **A graph G is P4-sparse if no set of five vertices of G induces more than one P4.**



Cographs: a class of graphs containing no P4s

P4-reducible graphs: a class of graphs such that every vertex belongs to at most one P4

Applications: scheduling, computational semantics, pattern recognition etc.



Definition For every graph G consider the graph C(G) returned by the following procedure:

```
Procedure Greedy(G);
{Input: an arbitray graph G;
 Output: a graph C(G)}
begin
    C(G) = G;
    while there exists a P4 in C(G) do
        pick an arbitrary P4 uvxy;
        pick z at randon in {u,y};
         C(G) = C(G)-{z};
    return(C(G))
end; {Greedy}
```

Theorem **For a graph G with no induced C5 the following statements are equivalent:**
(i) G is P4-sparse;
(ii) for every induced subgraph H of G, C(H) is unique up to isomorphism

Consider $G_1=(V_1,\emptyset)$ and $G_2=(V_2,E_2)$ $(V_1\cap V_2=\emptyset)$ with $V_2 = \{v\}\cup K\cup R$ such that

- $|K|=|V_1|+1\geq 2$
- K is a clique.
- Every vertex in R is adjacent to all the vertices in K and non-adjacent to v.
- There exists a vertex v' in K such that $N_{G_2}(v)=\{v'\}$ or $N_{G_2}(v)=K-\{v'\}$.

Choose a bijection $f: V_1 \to K-\{v'\}$ and define

$$G_1 \circledcirc G_2=(V_1\cup V_2, E_2\cup E')$$

with

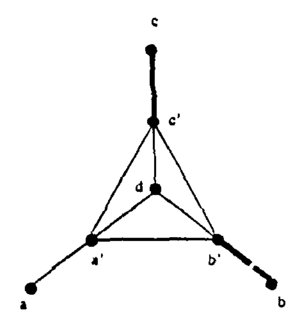
$$E' = \begin{cases} \{xf(x) \mid x\in V_1\} & \text{whenever } N_{G_2}(v)=\{v'\} \\ \{xz \mid x\in V_1,\ z\in K-\{f(x)\}\} & \text{whenever } N_{G_2}(v)=K-\{v'\} \end{cases}$$

Theorem *G is a P4-sparse graph if, and only if, G is obtained from single-vertex graphs by a finite sequence of operations* $\circledcirc$, $\oplus$, $\circledcirc$. □

```
Procedure Build_tree(G);
{Input: a P4-sparse graph G=(V,E);
 Output: the ps-tree T(G) corresponding to G.}
 begin
   if M = 1 then
      return the tree T(G) consisting of the unique vertex of G;
   if G (G̅) is disconnected then begin
      let G1, G2,..., Gp (p≥2) be the components of G (G̅);
      let T1, T2, ...,Tp be the corresponding ps-trees rooted at r1, r2, ..., rp;
      return the tree T(G) obtained by adding r1, r2,..., rp as children of a node
      labelled 0 (1);
      end
    else begin {now both G and G̅ are connected}
      write G = G1 ⊕ G2
      let T1, T2 be the corresponding ps-trees rooted at r1 and r2;
      return the tree T(G) obtained by adding r1, r2 as children of a node labelled 2
      end
end; {Build_tree}
```

An example...

G



G_1 $(\{b,c\},\Phi)$

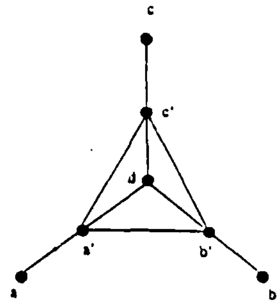
G_2 $(\{a\}\cup\{a',b',c'\}\cup\{d\},\{aa',a'b',a'c',b'c',a'd,b'd,c'd\})$

v v'

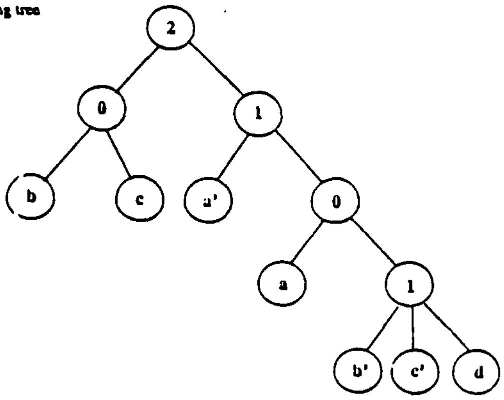
K R

An example...

a P4–sparse graph



the corresponding tree

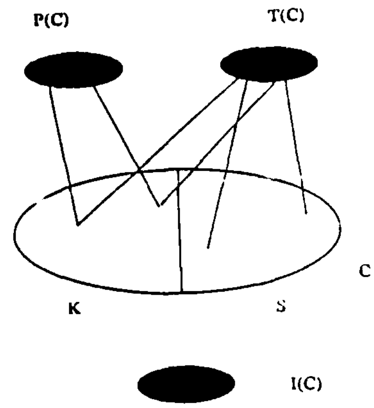


A set C of vertices of G is termed *regular* if it admits a partition into non-empty, disjoint sets K and S satisfying the following conditions:

(r1) $|K|=|S|\geq 2$, S stable, K a clique;

(r2) every vertex in V–C belongs to precisely one of the sets:
- $T(C)=\{x\in V-C \mid x$ adjacent to all the vertices in $C\}$;
- $I(C)=\{x\in V-C \mid x$ *non-adjacent to all the vertices in* $C\}$;
- $P(C)=\{x\in V-C \mid x$ adjacent to all the vertices in K and non-adjacent to all the vertices in $S\}$.

(r3) there exists a bijection $f:S\rightarrow K$ such that
- either $N(x)\cap K=\{f(x)\}$ for every x in S,
- or else $N(x)\cap K=K-\{f(x)\}$ for every x in S.



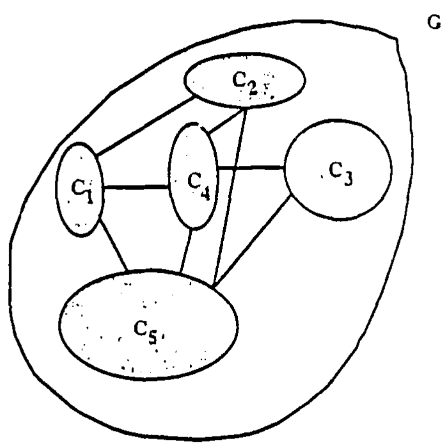
Let G = (V,E) be an *arbitrary* graph.

Fact (*Regularity is hereditary*)
Let C = (K,S,f) be a regular set in G and let Z be a subset of S with $|Z| < |S| - 2$. Then $C' = C - \{x,f(x) \mid x \in Z\}$ is regular

Fact (*Containment*)
Let C = (K,S,f) be regular. For every pair of distinct u, v in C with $u \neq f(v)$ and $v \neq f(u)$, the unique P4 containing u and v belongs to C

Fact (*Black hole property*)
A regular set is maximal if, and only if, every regular P4 containing a vertex in C is included in C

Fact (*Separation property*)
Two maximal regular sets coincide whenever they intersect



The "world " of regular sets

Given an arbitrary graph G construct a graph G^* as follows:

remove in every maximal regular set $C = (K,S,f)$ all the vertices in S except for an arbitrary one

Theorem For every graph G, the graph G^* is unique up to isomorphism

Theorem For an arbitrary graph G the following statements are equivalent:
(i) G is P4-sparse;
(ii) G^* is a cograph

Algorithm Recognize(G);
{Input: an arbitrary graph G;
Output: "yes" or "no" depending on whether or not G is P4–sparse}

Step 1. Find all maximal regular sets in G;

Step 2. Compute G^*;

Step 3. if G is a cograph then
return("yes")
else
return("no")

Step 4. Stop.

Our algorithm:

O(log n) EREW time using $O\left(\frac{n^2+mn}{\log n}\right)$ processors

What we do:

- recognize P4-sparse graphs;
- construct the corresponding tree

The Algorithm

- The EREW model of computation is assumed;
- G is an arbitrary graph represented by adjacency lists;
- for every vertex x, assign one processor to every entry on the adjacency list of x;
- the vertices are enumerated as $v_1, v_2, \ldots, v_n$ in a way that will be explained later;
- sets will be represented by their characteristic vector;
  - * computing the cardinality of a set takes O(log n) time using O(n/log n) processors;
  - * given sets S, S' of vertices of G, computing $S-S'$, $S \cup S'$, $S \cap S'$, as well as testing $S = \emptyset$, $S \subseteq S'$ takes O(log n) time using O(n/log n) processors;
- to compute N[x] we need O(log n) time and O(n/log n) processors.

Processor assignment:

- for every x of G, every entry on the adjacency list of x receives one processor;
- every edge $e_i (i=1,2,\ldots,m)$ receives $1+\lceil n/\log n \rceil$ processors

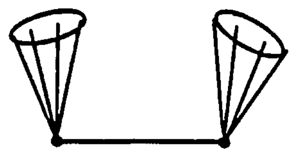
  $P(e_i, 0), P(e_i, 1), \ldots P(e_i, m)$

  Note: the total number of processors is bounded by

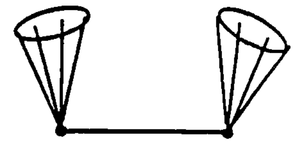
  $O(\frac{mn}{\log n})$

How do we find a regular set?

$N_{uw} = N[u]-N[w]$ $\quad$ $N_{wu} = N[w]-N[u]$



$N_{vw} = N[v]-N[w]$ $\quad$ $N_{wv} = N[w]-N[v]$



Fact The edge vw is the *midedge* of a regular P4 in G only if $|N_{vw}| = |N_{wv}| = 1$, and $uz \notin E$ w 'h u, z standing for the unique vertex in N_{vw} and N_{wv}, respectively

- For every edge e =vw the sets N_{vw} and N_{wv} can be computed in O(log n) time as follows:
  - N[v] will be broadcast to all $d_G(v)$ edges incident with v.
  - N[w] will be broadcast to all $d_G(w)$ edges incident with w

```
Procedure Find_Regular_P4s(G);
begin
  for every edge e_i={v,w} of G do in parallel begin
     N_vw <- N[v]-N[w];
     N_wv <- N[w]-N[v];
     if |N_vw|=|N_wv|=1 then {let N_vw={u}, N_wv={z}, U={u,v,w,z}}
        if uz ∉ E then begin
           for all the vertices x in V-{u,v,w,z} do in parallel
              if x ∉ T(U)∪P(U)∪I(U) then
                  some processor P(e_i,i(≠0)) writes a "1" in its own memory;
           if no "1" was written then P(e_i,0) does the following
                · remembers {u,v,w,z};
                · flags itself
        end {if}
  end {for}
end; {Find_Regular_P4s}
```

Fact *Procedure Find_Regular_P4s correctly computes the set of all the regular P_4s in G in O(log n) EREW time using $O(\frac{n^2+mn}{\log n})$ processors.*

More terminology...

regular P4

Assume $u = v_j$ and $z = v_k$ with $j < k$

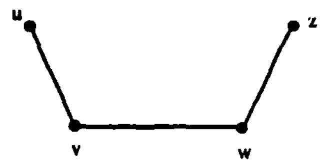
u is local "loser"

v is local "winner"

- Each flagged processor P(i) writes the identity of the local loser and winner into A[i] and B[i], respectively (A and B are one-dimensional arrays of m elements initialized to 0)
- Sort all non-zero entries of A and B and remove duplicates: this takes O(log n) time using O(m) processors
- Construct a bitvector L: bit i of L is set to 1 iff v_i is a local loser (This takes O(log n) time and O(n/log n) processors)

More terminology...

regular P4



- An endpoint u of a regular P4 is called a *global winner* if the bit corresponding to u in L is 0
- To record all the global winners we construct a bitvector W using the information in the array B; (This takes O(log n) time and O(n/log n) processors)
- W = W – L (this is the set of all global winners)
- Every flagged processor corresponding to a global winner is referred to as *essential*

```
Procedure Find_Winners_and_Losers(C),
begin
    A[1:m]←B[1:m]←0;
    L←W←0;
    for every flagged processor P(i) in parallel begin
        A[i]← local loser corresponding to e_i;
        B[i]← local winner corresponding to e_i
        end; {for}
    let A[1], A[2],...A[k] be the non-zero entries of A
    in sorted order with all duplicates removed;
    let B[1], B[2],...B[l] be the non-zero entries of B
    in sorted order with all duplicates removed;
    for all i←1 to k do in parallel
        set the A[i]-th bit of L to 1;
    for all i←1 to l do in parallel
        set the B[i]-th bit of W to 1.
    W←W–L; {find global winners}
    broadcast W to all the processors P(i);
    for every flagged processor P(i) in parallel
        if the local winner of e_i is in W then
            P(i) does the following:
                - remembers that its local winner is a global winner;
                - marks itself as "essential"
    return(L,W)
end; {Find_Winners_and_Losers}
```

Fact . *Procedure Find_Winners_and_Losers correctly computes the set of all the global winners and losers in O(log n) EREW time using* $O(\frac{mn}{\log n})$ *processors.*

```
Procedure Construct_SK(G);
begin
    let w_1, w_2, ..., w_p stand for the global winners;
    for i←1 to p do in parallel
        if processor P(i) is essential then begin
            processor P(i) sets to 1 the bit of S_i cooresponding to w_i;
            let P(i1), P(i2), ..., P(it_i) (1≤i≤p) be the
            essential processors whose local winner is w_i;
            for j←1 to t_i do in parallel
                processor P(ij) sets the k-th bit of S_i with
                v_k standing for its local loser;
            processor P(i1) broadcasts to P(i2), ..., P(it_i)
            the identity of the two midpoints it stores;
            for j←2 to t_i do in parallel
                processor P(ij) marks the midpoint it stores coinciding
                with one of the midpoints received;
            for j←1 to t_i do in parallel
                processor P(ij) sets to 1 the bit of K_i corresponding
                to its unmarked midpoint;
            r_i←|K_i|+|S_i|
            if |N(w_i)∩K_i|=1 then
                f_i(w_i)← the unique vertex in N(w_i)∩K_i
            else
                f_i(w_i)← the unique vertex in K_i–N(w_i)
        end; {if}
    return(SK(G))
end; {Construct_SK}
```

To summarize our previous discussion, we state the following result.

Fact *Procedure Construct_SK correctly computes the information in every SK[i] (1≤i≤p) in O(log n) time using* $O(\frac{n^2}{\log n})$ *processors in the EREW-PRAM model.* □

```
Procedure Recognize_P4sparse(G);
{Input: an arbitrary graph G,E) with |V|=n and |E|=m;
Output: "yes" or "no" depending on whether or not G is a P4-sparse graph:}
begin
    Find_Regular_P4s(G);
    Find_Winners_and_Losers(G);
    using the information contained in L construct the graph G*;
        if Cograph(G*) then
            return("yes");
    return("no")
end; {Recognize_P4sparse}
```

Theorem *Procedure Recognize_P4sparse correctly determines whether an arbitrary graph G=(V,E) with |V|=n and |E|=m is a* P_4*-sparse graph in O(log n) time using* $O(\frac{n^2+mn}{\log n})$ *processors in the EREW-PRAM model.*

Constructing the tree representation of P4–sparse graphs

- $T(G)$, *the cotree of the reduced graph* G^* *is available as a byproduct of* $Cograph(G^*)$
- for convenience we enumerate the maximal regular sets as

  $C_1 = (K_1, S_1, f_1),\ C_2 = (K_2, S_2, f_2),\ \dots,\ C_p = (K_p, S_p, f_p),$

- at the end of the successful recognition of a P4–sparse graph G, the relevant information about G is stored in the tuple $(T(G), SK(G))$

What is SK(G)??

- We can think of SK(G) as a 1–dimensional array such that S[i] contains the following information
  - characteristic vectors of K_i and S_i
  - the identity of the unique vertex w_i in S_i that b to G^*
  - the identity of $f_i(w_i)$
  - $r_i = |K_i| = |S_i|$

Let w , w , ..., w be the global winners as recorded in W

- To compute S_i every essential processor whose local winner is w_i sets the j–th bit of S_i, with j standing for its local loser

- To compute K_i we do the following
  - In O(log n) time identify the subset $P(i_1), P(i_2), \dots, P(i_{t_i})$ of essential processors whose local winner is $v_{W[i]}$
  - Processor $P(i_1)$ broadcasts to $P(i_2), \dots, P(i_{t_i})$ the identity of the midpoint it has remembered
  - Every processor $P(i_j)$ marks its own midpoint coinciding with the one received by broadcasting
  - Every processor $P(i_j)$ sets to 1 the bit of K_i corresponding to the unmarked midpoint it stores

```
Procedure Parallel_Build_ps_Tree(G);
{Input: a P4-sparse graph represented as (T(G),SK(G))
 Output: the corresponding ps-tree T(G), rooted at R;}
begin
   for every essential processor P(i) do in parallel begin
       create a 2-node β;
       create a 1-node γ;
       add γ as a child of β;
       add λ as a child of γ;
       if r_i=2 then begin
           add the unique vertex in S_i-{w_i} as a child of β;
           add f_i(w_i) as a child of γ
           end
       else begin
           create a 0-node α;
           add α as a child of β;
           add all vertices in S_i-{w_i} as children of α;
           if w_i is adjacent to f_i(w_i) then
              add f_i(w_i) as a child of γ
           else
              add all vertices in K_i-f({w_i}) as children of γ
           end; {if}
       if d(λ')≠|N(w_i)∩K_i|+1 then
           add β as a child of λ'
       else begin
           add β as a child of p(λ');
           delete λ'
           end {if}
   end; {for}
   if d(R)=1 then R←unique child of R;
   return(T(G))
end; {Build_ps_Tree}
```

Theorem *Procedure Parallel_Build_ps_Tree correctly constructs the ps-tree of a* P_4*–sparse graph* $G=(V,E)$ *with* $|V|=n$ *and* $|E|=m$ *in* $O(\log n)$ *EREW time using* $O(\frac{n}{\log n})$ *processors.*

Vertex-switching reconstruction and pseudosimilarity

Prof. Mark Ellingham
Department of Mathematics
Vanderbilt University

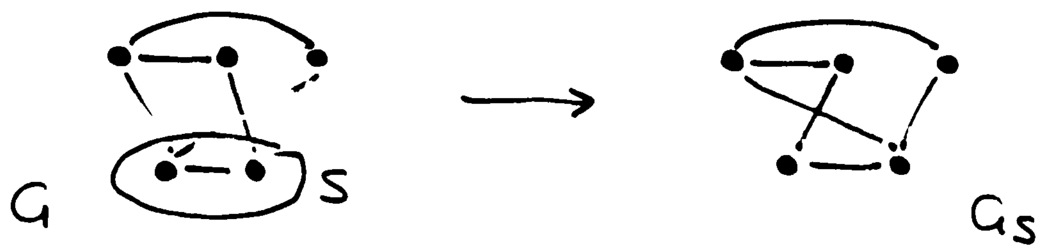
Recent Results on Vertex-Switching Reconstruction

Mark Ellingham
Vanderbilt University

Defn: Let S be a set of vertices in graph G. Then $\bar{S}$ denotes $V(G)-S$ (vertices not in S).

The S-switching G_S of G is obtained by
(1) deleting all present edges between S and $\bar{S}$
(2) adding all absent edges between S and $\bar{S}$

Ex:

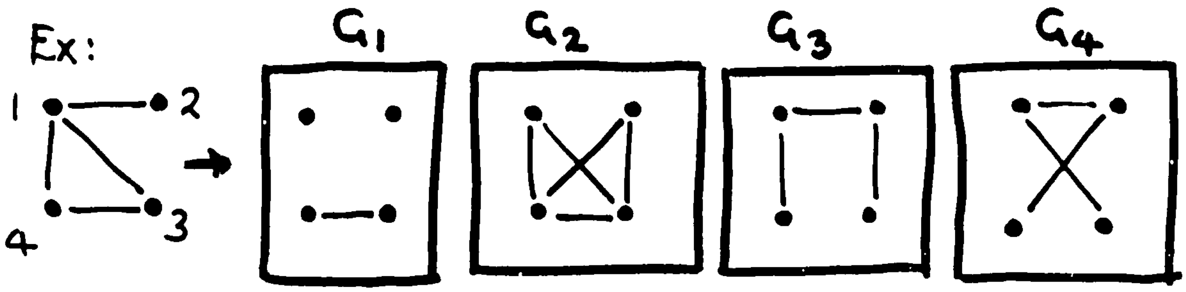


Notation: $G_v = G_{\{v\}}$, a vertex-switching
$G_{uv} = (G_u)_v$

Notes: $G_{\bar{S}} = G_S$, $G_{vv} = G$, $G_{uv} = G_{vu}$.

Defn: The vertex-switching deck $D_{VS}(G)$ is the collection of vertex-switchings of G.

Ex:



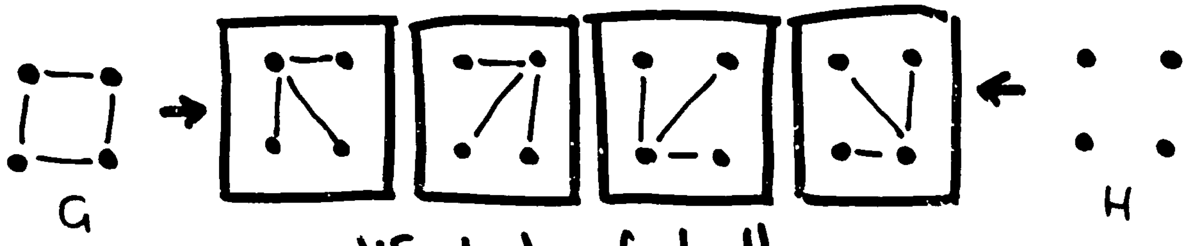
VS deck of G
(no labels on vertices)

Defn: H is a VS-reconstruction of G if

$$D_{VS}(H) = D_{VS}(G)$$

- G is VS reconstructible if every VS reconstruction of G is isomorphic to G.

Ex: A non-VS reconstructible pair:



VS deck of both

Several other such pairs on 4 vertices exist.

VS Reconstruction Conjecture (Stanley 1985): Any graph with $n \neq 4$ vertices is VS reconstructible

Theorem (Stanley 1985): An n-vertex graph with n not divisible by 4 is VS reconstructible.

Open Question: What about graphs with 12, 16, 20, ... vertices? (For 8 vertices, conjecture true by computer testing.)

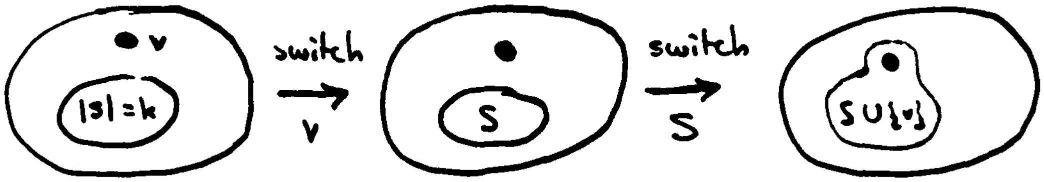
Alternative proof (Krasikov & Roditty)

- uses counting methods.

Notation: $X_k(G \to F)$ = number of k-switchings of G (i.e. G_S, $|S| = k$) which are isomorphic to F.

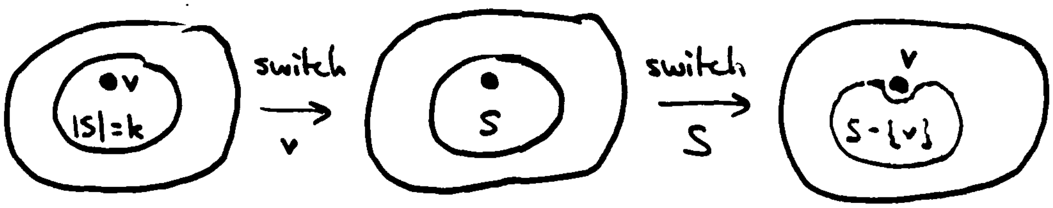
Examine k-switchings of cards in VS deck, i.e. k-switchings of 1-switchings of G. Two cases:

(1)



$(k+1)$-switching

(2)



$(k-1)$-switching

In fact

$$\sum_{J \in D_{VS}(G)} X_k(J \to F) = (k+1) X_{k+1}(G \to F) + (n-k+1) X_{k-1}(G \to F) \qquad (A)$$

But if $D_{VS}(G) = D_{VS}(H)$ same equation holds with G replaced by H, so if, for given F, we define

$$\delta_k = X_k(G \to F) - X_k(H \to F)$$

we get

$$(k+1)\,\delta_{k+1} + (n-k+1)\,\delta_{k-1} = 0$$

(subtract (A) for H from (A) for G).

In particular, if $F = G$ we have

$$\delta_k = X_k(G \to G) - X_k(H \to G)$$

and we get

$$(k+1)\,\delta_{k+1} + (n-k+1)\,\delta_k = 0 \qquad (B)$$

where, if $G \not\cong H$,

$$\left.\begin{array}{l} \delta_0 = 1 - 0 = 1 \\ \delta_1 = 0 \qquad \text{since } G, H \text{ have same} \\ \qquad\qquad \text{VS deck, i.e. same 1-switchings.} \end{array}\right\} (C)$$

Solving (B) with initial conditions (C) gives

$$\left.\begin{array}{l} \delta_{2i} = (-1)^i \binom{n/2}{i} \\ \delta_{2i+1} = 0 \end{array}\right\} (D)$$

But also, since $G_S = G_{\bar{S}}$, must have $\delta_k = \delta_{n-k}$.

However,

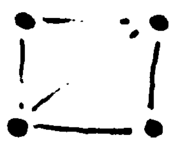
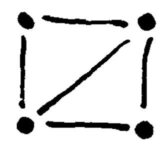
- if n odd $\quad \delta_0 = 1 \neq \delta_n = 0$
- if $n \equiv 2 \pmod 4$ $\quad \delta_0 = 1 \neq \delta_n = -1$

So we can only have $D_{VS}(G) = D_{VS}(H)$ but $G \not\cong H$ if $n \equiv 0 \pmod 4$. □

Important: For non-VS reconstructible G, (D) says that $\delta_k > 0$, implying that $X_k(G \to G) > 0$, if k is divisible by 4. Thus for any k divisible by 4, $0 \le k \le n$, G has k-switching $G_S \cong G$.

Reconstructing subgraph numbers from VS deck
(Stanley, for edges. ME & Royle / Krasikov & Rodity in general)

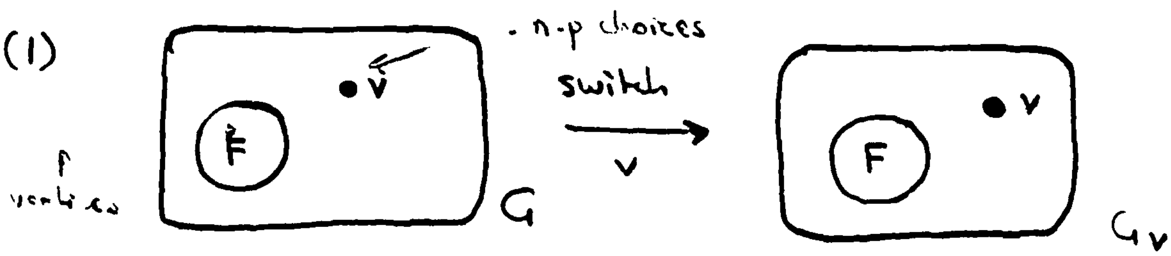
Defn: An induced subgraph of graph G contains all edges incident with a given vertex set.

Ex:  induced  not induced

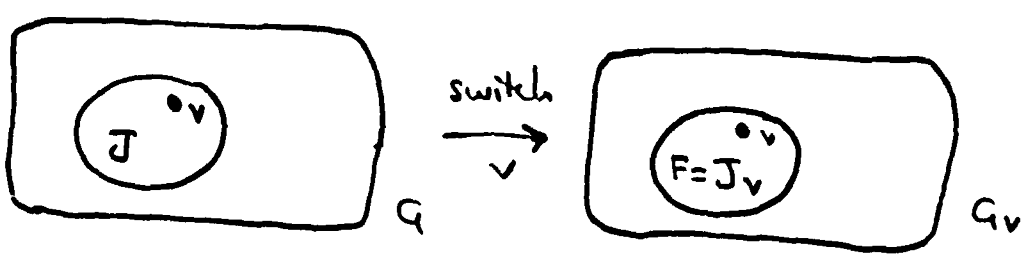
Let $i(F, G)$ be number of induced subgraphs of G which are isomorphic to F. Want to find $i(F, G)$ from VS deck.

How can F occur in VS deck? Two cases:

(1)



(2)



So for p-vertex F get equation

$$\sum_{C \in D_{VS}(G)} i(F,C) = (n-p)\, i(F,G) + \sum_{\substack{p\text{-vertex} \\ \text{unlabelled} \\ J\text{'s}}} X_1(J \to F)\, i(J,G)$$

By taking all such equations for switching class of graphs, get system of linear equations, try to solve for $i(F,G)$'s.

Ex: Switching class $\Big\{$  C_3,  $\bar{P}_3$ $\Big\}$

Get equations

C_3's in deck $= (n-3)\, i(C_3,G) + i(\bar{P}_3, G)$

$\bar{P}_3$'s in deck $= (n-3)\, i(\bar{P}_3,G) + 3i(C_3,G) + 2i(\bar{P}_3,G)$

i.e.

$$\begin{pmatrix} n-3 & 1 \\ 3 & n-1 \end{pmatrix} \begin{pmatrix} i(C_3,G) \\ i(\bar{P}_3,G) \end{pmatrix} = \begin{pmatrix} C_3\text{'s in deck} \\ \bar{P}_3\text{'s in deck} \end{pmatrix}$$

Can solve provided $(n-3)(n-1)-3 \neq 0$, i.e. $n \neq 0, 4$.

Theorem: For n-vertex G with $n \equiv 0 \pmod 4$ and p-vertex F, can VS reconstruct $i(F,G)$ if $n > 2p$.

Corollary: For $n \neq 4$ can VS reconstruct number of edges and vertex degrees.

VS Reconstruction by structural means

Assume $D_{VS}(G) = D_{VS}(H)$, $G \not\cong H$.

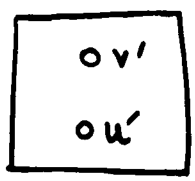
Lemma (Krasikov & Roditty): For every vertex v of G there exists u such that

(i) $G_{vu} \cong H$;

(ii) $\{v,u\}$ joined by exactly $n-2$ edges to $\overline{\{v,u\}}$;

(iii) v and u have a common neighbour in G.

Proof: (i)



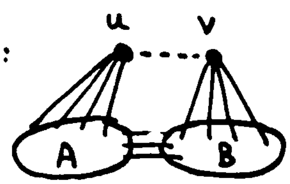
C card

$C \cong G_v$ | $C \cong H_{u''}$
$v' \leftrightarrow v$ | $v' \leftrightarrow v''$
$u' \leftrightarrow u$ | $u' \leftrightarrow u''$

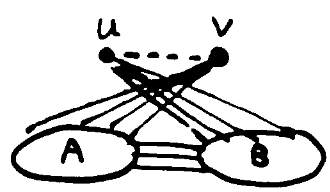
Now $G_{vu} = (G_v)_u \cong C_{u'} \cong (H_{u''})_{u''} = H$.

(ii) $G_{vu} \cong H$ has exactly same number of edges as G. So G has exactly half of possible $2(n-2)$ edges from $\{v,u\}$ to $\overline{\{v,u\}}$.

(iii) If not:



G



$G_{vu} \cong H$

But then $G_{vu} \cong G$, so $H \cong G$, contradiction.

VS Reconstruction Results for $n \equiv 0 \pmod{4}$

- disconnected graphs (Krasikov)
  Used structural lemma.

- graphs with $n \binom{n-1}{\Delta} < 2^{n/2-2}$ (Δ = maximum degree) (Krasikov)
  Used $\delta_{4j} > 0$, counting arguments.

- regular graphs (ME & Royle)
  Used structural arguments. Simple, but harder than for vertex deletion reconstruction.

- triangle-free graphs (ME & Royle)
  Recognition: used fact that $i(C_3, G)$ is reconstructible.
  Reconstruction: used structural lemma, regular graphs result.

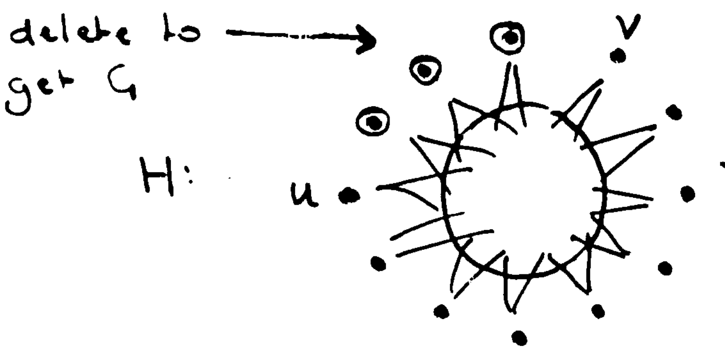
Vertex switching pseudosimilarity

Defn: u, v similar if some automorphism maps u to v

quasisimilar if $G-u \cong G-v$

pseudosimilar if quasisimilar, not similar

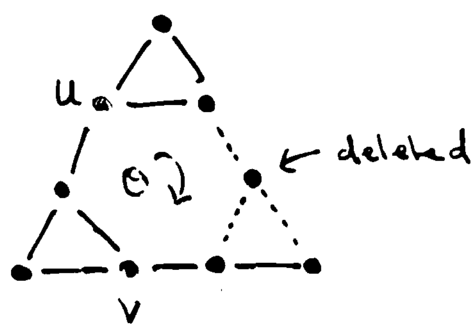
Theorem (Godsil & Kocay): All pairs of quasi-similar vertices in finite graphs arise from following construction:



automorphism Θ of H

(green vertices are an orbit of Θ)

Ex:



Harary & Palmer's original example

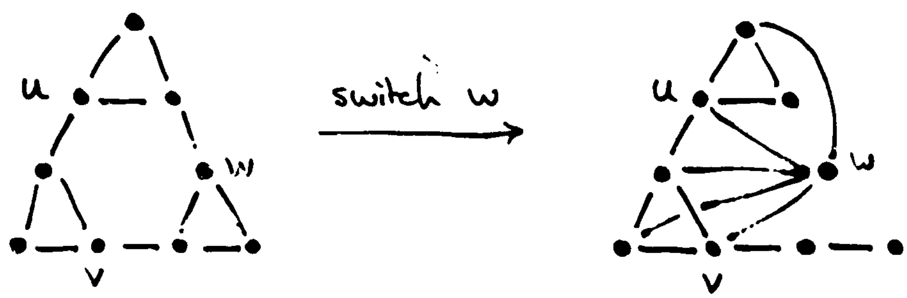
Defn: u, v VS quasisimilar if $G_u \cong G_v$

VS pseudosimilar if VS quasisimilar, not similar

Theorem (ME): For finite graphs, all occurrences of VS quasisimilar vertices arise from two constructions:

(i) analogous to Godsil & Kocay's construction

Ex:



u, v VS pseudosimilar

(ii) funny construction involving switching alternate vertices along orbits of automorphism θ of graph H.

Note: Proof of Theorem involved characterising all situations where $G_S \cong G$ for some set of vertices S. Used this because if $G_u \cong G_v$ then $\underset{\overset{\uparrow}{S}}{(G_u)_{\{u,v\}}} = G_v \cong G_u$.

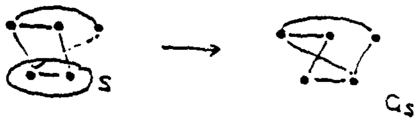
Characterisation of when $G_S \cong G$ has other possible implications. Know that for nonreconstructible G, must be sets S of size divisible by 4 with $G_S \cong G$.

Recent Results on Vertex-Switching Reconstruction

Mark Ellingham
Vanderbilt University

Defn: Let S be a set of vertices in graph G. Then $\bar{S}$ denotes $V(G)-S$ (vertices not in S).
The S-switching G_S of G is obtained by
(1) deleting all present edges between S and $\bar{S}$
(2) adding all absent edges between S and $\bar{S}$

Ex:

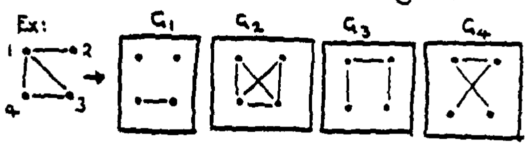


Notation: $G_v = G_{\{v\}}$, a vertex-switching
$G_{uv} = (G_u)_v$

Notes: $G_{\bar{S}} = G_S$, $G_{vv} = G$, $G_{uv} = G_{vu}$.

Defn: The vertex-switching deck $D_{VS}(G)$ is the collection of vertex-switchings of G.

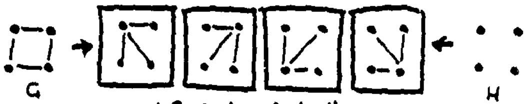
Ex:



VS deck of G
(no labels on vertices)

Defn: H is a VS-reconstruction of G if
$D_{VS}(H) = D_{VS}(G)$
G is VS reconstructible if every VS reconstruction of G is isomorphic to G.

Ex: A non-VS reconstructible pair:



VS deck of both

Several other such pairs on 4 vertices exist.

VS Reconstruction Conjecture (Stanley 1985):
Any graph with $n \neq 4$ vertices is VS reconstructible

Theorem (Stanley 1985): An n-vertex graph with n not divisible by 4 is VS reconstructible.

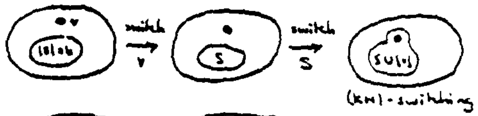
Open Question: What about graphs with 12, 16, 20, ... vertices? (For 8 vertices, conjecture true by computer testing.)

Alternative proof (Krasikov & Roditty)
- uses counting methods.

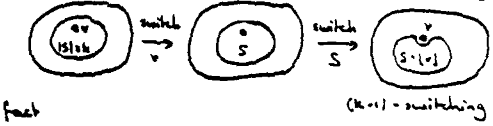
Notation: $X_k(G \to F)$ = number of k-switchings of G (i.e. G_S, $|S| = k$) which are isomorphic to F.

Examine k-switchings of cards in VS deck, i.e. k-switchings of 1-switchings of G. Two cases:

(1)



(2)



In fact

$$\sum_{J \in D_{VS}(G)} X_k(J \to F) = (k+1) X_{k+1}(G \to F) + (n-k+1) X_{k-1}(G \to F) \qquad (A)$$

But if $D_{VS}(G) = D_{VS}(H)$ same equation holds with G replaced by H, so if, for given F, we define

$$\delta_k = X_k(G \to F) - X_k(H \to F)$$

we get

$$(k+1)\,\delta_{k+1} + (n-k+1)\,\delta_{k-1} = 0$$

(subtract (A) for H from (A) for G).

WVS-7 (5)

In particular, if $F = G$ we have

$$\delta_k = X_k(G \to G) - X_k(H \to G)$$

and we get

$$(k+1)\,\delta_{k+1} + (n-k+1)\,\delta_k = 0 \qquad (B)$$

where, if $G \not\cong H$,

$$\left.\begin{array}{ll} \delta_0 = 1 - 0 = 1 & \\ \delta_1 = 0 & \text{since } G, H \text{ have same VS deck, i.e. same 1-switchings} \end{array}\right\} (C)$$

Solving (B) with initial conditions (C) gives

$$\left.\begin{array}{l} \delta_{2i} = (-1)^i \binom{n/2}{i} \\ \delta_{2i+1} = 0 \end{array}\right\} (D)$$

But also, since $G_S = G_{\bar{S}}$, must have $\delta_k = \delta_{n-k}$.

However,

- if n odd $\quad \delta_0 = 1 \neq \delta_n = 0$
- if $n \equiv 2 \pmod 4$ $\quad \delta_0 = 1 \neq \delta_n = -1$

So we can only have $D_{VS}(G) = D_{VS}(H)$ but $G \not\cong H$ if $n \equiv 0 \pmod 4$. □

Important: For non-VS reconstructible G, (D) says that $\delta_k > 0$, implying that $X_k(G \to G) > 0$, if k is divisible by 4. Thus for any k divisible by 4, $0 \le k \le n$, G has k-switching $G_S \cong G$.

(6)

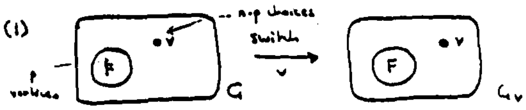
Reconstructing subgraph numbers from VS deck (Stanley, for edges. McK & Royle / Krasikov & Roditty in general)

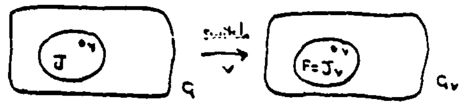
Defn: An induced subgraph of graph G contains all edges incident with a given vertex set.

Ex:  induced  not induced

Let $i(F,G)$ be number of induced subgraphs of G which are isomorphic to F. Want to find $i(F,G)$ from VS deck.

How can F occur in VS deck? Two cases:

(1) 

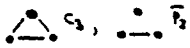
(2) 

(7)

So for p-vertex F get equation

$$\sum_{\sim G \in D_{VS}(G)} i(F,C) = (n-p)\, i(F,G) + \sum_{\substack{p\text{-vertex}\\ \text{unlabelled}\\ J\text{'s}}} X_1(J \to F)\, i(J,G)$$

By taking all such equations for switching class of graphs, get system of linear equations, try to solve for $i(F,G)$'s.

Ex: Switching class $\{$  C_3, $\bar{P}_3$ $\}$

Get equations

C_3's in deck $= (n-3)\, i(C_3,G) + i(\bar{P}_3, G)$

$\bar{P}_3$'s in deck $= (n-3)\, i(\bar{P}_3,G) + 3 i(C_3,G) + 2 i(\bar{P}_3,G)$

i.e.

$$\begin{pmatrix} n-3 & 1 \\ 3 & n-1 \end{pmatrix} \begin{pmatrix} i(C_3,G) \\ i(\bar{P}_3,G) \end{pmatrix} = \begin{pmatrix} C_3\text{'s in deck} \\ \bar{P}_3\text{'s in deck} \end{pmatrix}$$

Can solve provided $(n-3)(n-1) - 3 \neq 0$, i.e. $n \neq 0, 4$.

Theorem: For n-vertex G with $n \equiv 0 \pmod 4$ and p-vertex F, can VS reconstruct $i(F,G)$ if $n > 2p$.

Corollary: For $n \neq 4$ can VS reconstruct number of edges and vertex degrees.

WVS-7 (8)

VS Reconstruction by structural means

Assume $D_{VS}(G) = D_{VS}(H)$, $G \not\cong H$.

Lemma (Krasikov & Roditty): For every vertex v of G there exists u such that

(i) $G_{vu} \cong H$;
(ii) $\{v,u\}$ joined by exactly $n-2$ edges to $\overline{\{v,u\}}$;
(iii) v and u have a common neighbour in G.

Proof: (i)  C card

$C \cong G_v$ $\quad$ $C \cong H_{u''}$

$v' \leftrightarrow v$ $\quad$ $v' \leftrightarrow v''$

$u' \leftrightarrow u$ $\quad$ $u' \leftrightarrow u''$

Now $G_{vu} = (G_v)_u \cong C_{u'} \cong (H_{u''})_{u''} = H$.

(ii) $G_{vu} \cong H$ has exactly same number of edges as G. So G has exactly half of possible $2(n-2)$ edges from $\{v,u\}$ to $\overline{\{v,u\}}$.

(iii) If not:  

But then $G_{vu} \cong G$, so $H \cong G$, contradiction.

VS Reconstruction Results for $n \equiv 0 \pmod 4$

- disconnected graphs (Krasikov)
  Used structural lemma.

- graphs with $n\binom{n-1}{\Delta} < 2^{n/2-2}$ (Δ = maximum degree) (Krasikov)
  Used $\delta_{aj} > 0$, counting arguments.

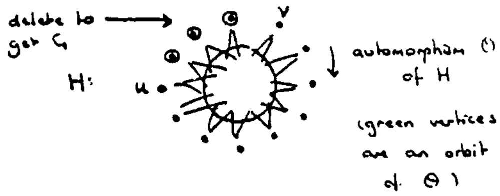
- regular graphs (ME & Royle)
  Used structural arguments. Simple, but harder than for vertex deletion reconstruction.

- triangle-free graphs (ME & Royle)
  Recognition: used fact that $i(C_3, G)$ is reconstructible.
  Reconstruction: used structural lemma, regular graphs result.

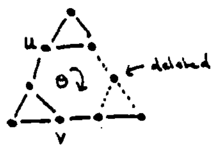
Vertex switching pseudosimilarity

Defn: u, v similar if some automorphism maps u to v
quasisimilar if $G-u \cong G-v$
pseudosimilar if quasisimilar, not similar

Theorem (Godsil & Kocay): All pairs of quasi-similar vertices in finite graphs arise from following construction:



Ex:



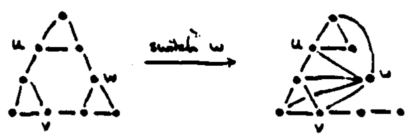
Harary & Palmer's original example

Defn: u, v VS quasisimilar if $G_u \cong G_v$
VS pseudosimilar if VS quasisimilar, not similar

Theorem (ME): For finite graphs, all occurrences of VS quasisimilar vertices arise from <u>two</u> constructions:

(i) analogous to Godsil & Kocay's construction

Ex:



u, v VS pseudosimilar

(ii) funny construction involving switching alternate vertices along orbits of automorphism θ of graph H.

Note: Proof of Theorem involved characterising all situations where $G_S \cong G$ for some set of vertices S. Used this because if $G_u \cong G_v$ then $\underbrace{(G_u)_{\{u,v\}}}_{S} = G_v \cong G_u$.

Characterisation of when $G_S \cong G$ has other possible implications. Know that for nonreconstructible G, must be sets S of size divisible by 4 with $G_S \cong G$.